Whittaker Functions on Metaplectic Groups

by

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Abstract

The theory of Whittaker functions is of crucial importance in the classical study of automorphic forms on adele groups. Motivated by the appearance of Whittaker functions for covers of reductive groups in the theory of multiple Dirichlet series, we provide a study of Whittaker functions on metaplectic covers of reductive groups over local fields.

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Contents

Chapter 1

Introduction

1.1 Introduction

Let G be a simply connected Chevalley group over a non-archimedean local field, and let U^- be opposite group to the unipotent radical of a Borel subgroup B. There are important representation-theoretic quantities that can be expressed as an integral over U^-, for example intertwining operators and Whittaker functions. The major aim of this thesis, comprising the results of [31], is the study and evaluation of such integrals.

Specifically, for χ a character of B trivial on the maximal compact subgroup K of G, define the function $\phi_K : G \longrightarrow \mathbb{C}$ by

$$\phi_K(bk) = (\delta^{1/2}\chi)(b)$$

for $b \in B$ and $k \in K$, where $\delta^{1/2}$ indicates the modular quasicharacter of B.

Then ϕ_K is a K-invariant vector in the principal series representation $I(\chi) = \mathrm{Ind}_B^G \delta^{1/2}\chi$. A formula for the (unique up to scaling) intertwining operator T between $I(\chi)$ and $I(\chi^{w_0})$ is given by

$$Tf(g) = \int_{U^-} f(uw_0g)du.$$

Here w_0 is the long element of the Weyl group, which acts on the set of principal series representations. It is most important to take $f = \phi_K$ and $g = 1$ in the above integral.

If we let ψ be a character of U^-, then the Whittaker function for the representation

$I(\chi)$ can be calculated by the integral

$$W(g) = \int_{U^-} \phi_K(ug)\psi(u)du.$$

The above integrals have direct analogues when one wishes to study the representation theory of central extensions $\widetilde{G}$ of G by a finite cyclic group μ_n. We shall refer to such groups as metaplectic groups, as opposed to the more restrictive notion of a metaplectic group as referring specifically to the double cover of the symplectic group. We shall review what we require from the representation theory of such groups in Section 3.2. Despite the restriction to the reductive case in the beginning of the introduction, our work will always hold more generally for the case of a metaplectic group, and we shall work in this generality throughout.

To each reduced decomposition $w_0 = s_{i_1} \dots s_{i_N}$ of the long element of the Weyl group into a product of simple reflections, we produce a decomposition of U^- into cells $C_{\mathbf{m}}$ indexed by N-tuples $\mathbf{m} = (m_1 \dots, m_N)$ of natural numbers, by producing an explicit version of the Iwasawa decomposition. The concept of realising a crystal combinatorially as a set of subvarieties of a unipotent radical dates back to Lusztig [29]. The decomposition we consider turns out to be equivalent to that of [20, Proposition 4.1]. However the approach taken in this paper is independent of that of Kamnitzer. The cells $C_{\mathbf{m}}$ have the property that the function ϕ_K is constant on each cell, so writing

$$\int_{U^-} = \sum_{\mathbf{m}} \int_{C_{\mathbf{m}}}$$

yields a combinatorial sum for these integrals we are studying, which becomes amenable to explicit calculation. In this manner, we obtain a method allowing us to evaluate our target family of integrals as a combinatorial sum over a crystal.

Indeed, there is a natural bijection between our collection of cells and the elements of the canonical basis $B(-\infty)$ of $U_q(\mathfrak{n}^+)$, the positive part of the quantised universal enveloping algebra. This connection with the combinatorics of crystals is studied in Section 3.4. In particular, Theorem 3.4.2 provides an explicit identification between the parametrisation of our cells and Lusztig's parametrisation of the canonical basis [28]. At the same time, we are able to relate our cell decomposition with previously constructed geometric models of crystals, in particular the realisation of the crystal in terms of Mirkovic-Vilonen cycles in the affine Grassmannian as in [8].

In Section 3.3, we are able to evaluate the integral for the intertwining operator in full generality, proving a metaplectic version of the Gindikin-Karpelevich Formula; this is the content of Theorem 3.3.4, itself a generalisation of [22, Proposition I.2.4].

In the case of Whittaker functions, to achieve explicit results, we restrict ourselves to working in type A with a particular choice of long word decomposition. We are then able to compute in Theorem 3.5.6 the metaplectic Whittaker function as a weighted sum over a crystal. We note that this weighted sum agrees exactly with the prime-power supported coefficients of a Weyl group Multiple Dirichlet Series, which we will refer to as the 'p-part'. These multiple Dirichlet series are certain Dirichlet series in several complex variables, satisfying a set of functional equations indexed by a Weyl group.

Weyl group multiple Dirichlet series were first introduced in [9]. These are global objects built from local components, namely their p-parts for each prime p. A combinatorial description of the p-part of such a series as a weighted sum over Gelfand-Tsetlin patterns is given in [10]. We give a full description of these coefficients at the beginning of Section 3.5. In fact this particular description is given an alternative description in terms of paths in a crystal graph in [10], which gives a natural interpretation of the integers $e_{i,j}$ that appear in the formulae.

This particular description was obtained in [10] by considering Whittaker coefficients of global metaplectic Eisenstein series attached to a minimal parabolic subgroup. The familiar unfolding technique may be applied in this metaplectic setting to show that the local components of Fourier coefficients of such an adelic metaplectic Eisenstein series are indeed given by a local Whittaker function. There is some subtlety with respect to the fact that multiple Dirichlet series do not admit an Euler product, but instead have coefficients which satisfy a weaker property of twisted multiplicativity.

Viewed in this regard, our results can be considered as an alternative approach to the combinatorial formulae appearing in [10]. In fact, in section 3.5, we detail our calculation in type A for a standard type of long word, exactly producing the combinatorial formulae of [10] in Theorem 3.5.6.

Because of the generality of our approach, which in principle can be used for arbitrary root systems and long word decompositions, our method may be viewed as providing a recipe for writing these Whittaker functions in the form of a generating function indexed by a crystal, as well as providing some understanding as to why the combinatorics of crystal graphs make an appearance in this field.

Independently, there has been an alternative calculation of the metaplectic Whittaker function in type A by Chinta and Offen [17]. This result shows that the spherical Whittaker function agrees with the p-parts of Weyl group multiple Dirichlet series using a different definition of the p-part, due to Chinta and Gunnells [16]. The results of this paper, combined with the work of Chinta and Offen provide a resolution of the question of proving that these two differing definitions of multiple Dirichlet series agree, which hitherto had been an open problem.

In order to carry out the aforementioned program, one needs some understanding of the structure of metaplectic groups and their principal series representations. The second goal of this thesis is to develop this theory, and consists of the work appearing in [32].

Let F be a non-archimedean local field with ring of integers O_F and assume that G is a split reductive group over F that arises by base extension from a smooth reductive group scheme $\mathbf{G}$ over O_F. Let n be a positive integer such that $2n$ is coprime to the residue characteristic of F and that $F^\times$ contains $2n$ distinct $2n$-th roots of unity. The object of this paper is to study the principal series representations of a group $\widetilde{G}$ which arises as a central extension of (the F points of) G by the cyclic group μ_n of order n. This means that there is an exact sequence of topological groups

$$1 \to \mu_n \to \widetilde{G} \to G \to 1$$

with the kernel μ_n lying inside the centre of $\widetilde{G}$. The topology on G induces the structure of a topological group on $\widetilde{G}$. We will use p to denote the natural projection map $\widetilde{G} \to G$.

This metaplectic group $\widetilde{G}$ is a locally compact, totally disconnected topological group. Following the example of reductive case, we study the simplest family of representations, namely those which are induced from the inverse image in $\widetilde{G}$ of a Borel subgroup of G. Such representations have been studied in the literature for particular classes of groups. Kazhdan and Patterson [22] have a detailed study in the case of $G = GL_n$, while Savin [39] has considered the case of G simply laced and simply connected. The double cover of a general simply connected algebraic group has also been considered by Loke and Savin [25]. This paper, in developing a theory in greater generality, borrows heavily on the results and arguments from the above mentioned papers, as often the existing arguments in the literature can be generalised to the case of an arbitrary reductive group. In another direction, we mention the work of Weissman [43] where the representation theory of metaplectic non-split tori is studied.

We begin by giving an overview of the construction of the metaplectic group. In order to carry out our construction in the desired generality, we are forced to use a significant amount of the theory of these extensions in the semisimple simply connected case, after which we proceed to the general reductive case along the lines of the approach in [18].

Our study of the representation theory begins by focussing on the metaplectic torus and its representations, which govern a large part of the following theory. This metaplectic torus is no longer abelian, but its irreducible representations are finite dimensional by a version of the Stone-von Neumann theorem.

Following this, we construct the principal series representations for a metaplectic group by the familiar method of inducing from a Borel subgroup. The theory of Jacquet modules and intertwining operators between such representations is developed in this generality.

We then study the Hecke algebras of anti-genuine compactly supported locally constant function that are invariant on the left and the right with respect to an open compact subgroup. The two cases of interest to us are when this compact subgroup is taken to be the maximal compact subgroup $K = \mathbf{G}(O_F)$ or an Iwahori subgroup. In the former case, we present a metaplectic version of the Satake isomorphism in Theorem 4.2.1, while in the Iwahori case, we give a presentation of the corresponding Hecke algebra in terms of generators and relations, following Savin [38, 39].

With the study of the structure of these Hecke algebras, we propose a combinatorial definition of the dual group to a metaplectic group which extends the notion of a dual group to a reductive group. This dual group is always a reductive group, so unlike in the reductive case, a metaplectic group cannot be recovered from its dual group. We hope that this notion of a dual group will prove to be useful in order to bring the study of metaplectic groups under the umbrella covered by the Langlands functoriality conjectures. It is worth noting that the root datum for the dual group has also appeared in [28, §2.2.5] in relation to the quantum Frobenius morphism, and in [18] and [36] where these Hecke algebras are studied geometrically.

It is believed to be possible to develop this theory while working under the weaker assumption that F only contains n n-th roots of unity, though in order to achieve this, a large amount of extra complications in formulae is necessary.

1.2 Preliminaries

In this section, we set up notation for use in the sequel. As mentioned in the introduction, F will denote a non-archimedean local field, O_F its valuation ring, and k its residue field. Let q be the cardinality of k. We choose once and for all a uniformising element $\varpi \in O_F$ (i.e. a generator of the maximal ideal of O_F). We write v for the valuation on F and $|\cdot|$ for the norm, normalised such that $|x| = q^{-v(x)}$. At one point, in order to establish the connection with crystal graphs in Section 3.4, we shall relax our assumption on F to include the case where F is the field of Laurent series over an algebraically closed field. The rest of this thesis however is concerned with the case where F is a local field.

Let $\mathbf{G}$ be a split reductive group scheme over O_F. Throughout, our practice will be to use boldface letters to denote the group scheme and roman face letters to denote the corresponding group of F-points. Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ with unipotent radical $\mathbf{U}$, and let $\mathbf{T}$ be a maximal split torus contained in $\mathbf{B}$. Let $\mathbf{U}^-$ denote the opposite unipotent subgroup to $\mathbf{B}$, thus $\mathbf{U}^-$ is a unipotent subgroup of $\mathbf{G}$ such that the multiplication map $\mathbf{U}^- \times \mathbf{B} \to \mathbf{G}$ is an open immersion. We let $X = \mathrm{Hom}(\mathbf{T}, \mathbb{G}_\mathbf{m})$ and $Y = \mathrm{Hom}(\mathbb{G}_m, \mathbf{T})$ be the character and cocharacter groups of $\mathbf{T}$. There is a natural perfect pairing $\langle \cdot, \cdot \rangle : X \times Y \longrightarrow \mathbb{Z}$.

Let Φ be the set of roots of $\mathbf{G}$ relative to $\mathbf{T}$. Our choice of a Borel subgroup determines Φ^+ and Φ^-, a choice of positive and negative roots respectively as well as the finite index set I of simple roots. We denote by $\Phi^\vee$ the set of coroots and $\alpha \mapsto \alpha^\vee$ the natural bijection between Φ and $\Phi^\vee$. We use $\langle \cdot, \cdot \rangle : \Phi \times \Phi^\vee \longrightarrow \mathbb{Z}$ to denote the canonical pairing between Φ and $\Phi^\vee$, inherited from the perfect pairing between X and Y.

In the simply connected case which will be our focus in Chapter 3 we have $Y = \mathbb{Z}\Phi^\vee$, while $\mathbb{Z}\Phi$ is a finite index sublattice of X. We also will require the corresponding adjoint group G^{ad} and the cocharacter group Λ of the corresponding torus T^{ad}. This cocharacter group Λ naturally contains the coroot lattice Y as a finite index sublattice.

Let W be the Weyl group of the root system Φ. It is generated by simple reflections s_i for each $i \in I$. Let w_0 denote the element of longest length in W and denote that length by N. The notation $[N]$ will be used for the set $\{1, 2, \ldots, N\}$.

Given a root α, there is an associated morphism of group schemes $\varphi_\alpha : SL_2 \longrightarrow \mathbf{G}$. Accordingly, we define elements of G by $w_\alpha = \varphi_\alpha(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))$ and $e_\alpha(x) = \varphi_\alpha(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right))$. We will show in section 1.6 that the Weyl group and unipotent subgroups of G split in the central extension $\widetilde{G}$ (in the latter case canonically) and use the same notation for the corresponding lifts to $\widetilde{G}$. For any $x \in F$ and $\lambda \in Y$, we will denote the image in T of x under λ by x^λ.

We fix a positive integer n which shall be the degree of our cover. The assumption on n we work under is that $2n | q - 1$. This implies the condition that F contains $2n$ $2n$-th roots of unity that was mentioned in the introduction.

Our object of study is not G itself, but a central extension of $\widetilde{G}$ by the group μ_n of n-th roots of unity. The construction of such a group will occupy the greater part of Chapter 1. For any subgroup H of G, we will use the notation $\widetilde{H}$ to denote the inverse image of H in $\widetilde{G}$, it is a central extension of H by μ_n.

We require some knowledge of the existence and properties of the Hilbert symbol, which we shall now recap. These results concerning the Hilbert symbol are well-known, a reference for this material may be given by Serre's book [40, Chapter 14].

The Hilbert symbol is a bilinear map $(\cdot, \cdot) : F^\times \times F^\times \to \mu_n$ such that

$$(s,t)(t,s) = 1 = (t, -t) = (t, 1-t).$$

Due to our assumptions on n, we can calculate the Hilbert symbol via the equation

$$(s,t) = \overline{\left((-1)^{v(s)v(t)} \frac{s^{v(t)}}{t^{v(s)}} \right)^{\frac{q-1}{n}}}$$

where the bar indicates to take the image in the residue field. Particular special cases that we will make liberal use of throughout this paper without further comment are the identities $(-1, x) = 1$ and $(\varpi^a, \varpi^b) = 1$, both of which require $q - 1$ to be divisible by $2n$, as opposed to only being divisible by n.

A representation (π, V) of $\widetilde{G}$ is defined to be a vector space V equipped with a group homomorphism $\pi : \widetilde{G} \longrightarrow \mathrm{Aut}\,(V)$. We say that (π, V) is *smooth* if the stabiliser of every vector contains an open subgroup, and *admissible* if for every open compact subgroup K, the subspace V^K of vectors fixed by K is finite dimensional.

Fix a faithful character $\epsilon : \mu_n \longrightarrow \mathbb{C}^\times$. We will only have cause to consider representations of $\tilde{G}$ in which the central μ_n acts by ϵ. (If μ_n did not act faithfully on an irreducible representation, then this representation would factor through a smaller cover of G.) Such representations will be called *genuine.*

We always will assume our representations to be genuine, smooth and admissible, and denote the category of such representations by $\mathrm{Rep}\,(\widetilde{G})$.

For any subgroup B of a group A, we use $Z_A(B)$ (respectively $N_A(B)$) to denote the centraliser (respectively normaliser) of B in A.

Let ψ denote an additive character of F with conductor O_F. This means that ψ is a homomorphism from the additive group F to $\mathbb{C}^\times$ that is trivial when restricted to O_F and non-trivial when restricted to $\frac{1}{\varpi}O_F$.

We will encounter Gauss sums appearing in our work, so we shall pause to define them and recap their properties. We define the Gauss sum $g(a,b)$ by the following integral

$$g(a,b) = \int_{O_F^\times} (u,\varpi)^a \psi(\varpi^b u) du. \tag{1.1}$$

where our measure du is the additive Haar measure on F, normalised such that $O_F^\times$ has volume $q-1$. This choice of normalisation of the Haar measure ensures that this approach to defining the Gauss sum agrees precisely with the classical definition as a sum over a finite field. We choose this approach since these precise integrals will appear later in this work. The following proposition provides a list of standard properties of these Gauss sums, their proofs involve routine manipulation of the defining integral.

Proposition 1.2.1. *Gauss sums satisfy the following identities.*

1. *For all a and b, $\overline{g(a,b)} = (-1,\varpi)^a g(-a,b)$.*

2. *If $b < -1$, then $g(a,b) = 0$.*

3. *If $b \geq 0$, then $g(a,b) = q-1$ if a is divisible by n, and zero otherwise.*

4. *If n divides a, then $g(a,-1) = -1$.*

5. *If a is not divisible by n, then $|g(a,-1)| = q^{1/2}$.*

Finally, we make a note about our usage of the symbol $\prod$ in the noncommutative case:

Either the terms in the product will all commute, or we will write $\prod_{k=m}^{n} x_k$ for $x_m x_{m+1} \dots x_n$ and $\prod_{k=n}^{m} x_k$ for $x_n x_{n-1}, \dots x_m$ where $m \leq n$.

1.3 Central Extensions of Chevalley Groups

This section and the following two will be taken up with presenting three different constructions of a central extension of G by μ_n. These constructions vary in the level of generality they provide, and our expositions vary in the level of detail according to the level of technical background utilised in the construction. It will prove to be useful to have knowledge given to us by understanding each of these constructions.

We begin by presenting the major features of the construction of the central extension where our group $\mathbf{G}$ is split and simply-connected. The results presented in this section are due to Steinberg [42]. This particular approach will prove to be useful to consider when it comes to our calculations in Chapter 3.

Since the group G is equal to its commutator subgroup, it admits a universal central extension E [35]. Thus we have a short exact sequence of groups

$$1 \to A \to E \to G \to 1$$

with A in the centre of E. Steinberg gives a presentation of E in terms of generators and relations, which we now quote:

Theorem 1.3.1 (Theorem 10, [42]). *The group E is generated by symbols $e_\alpha(x)$ where $\alpha \in \Phi$ and $x \in F$, subject to the relations*

$$e_\alpha(x)e_\alpha(y) = e_\alpha(x+y)$$

$$w_\alpha(x)e_\alpha(y)w_\alpha(-x) = e_{-\alpha}(-x^{-2}y)$$

where $w_\alpha(x) = e_\alpha(x)e_{-\alpha}(-x^{-1})e_\alpha(x)$, and

$$e_\alpha(x)e_\beta(y) = \Big[\prod_{\substack{i,j \in \mathbb{Z}^+ \\ i\alpha + j\beta = \gamma \in \Phi}} e_\gamma(c_{i,j,\alpha,\beta}x^i y^j)\Big]e_\beta(y)e_\alpha(x) \tag{1.1}$$

for all $x, y \in F$ and $\alpha, \beta \in \Phi$ with $\alpha + \beta \neq 0$, where $c_{i,j,\alpha,\beta}$ is a fixed collection of integers, *completely determined by the root system* Φ.

Note that the product in (1.1) is a product of commuting terms, so there is no possible ambiguity present with respect to order of multiplication.

Matsumoto [30] gave a computation of the kernel A, and showed it to be equal to $K_2(F)$ except in type C in which case $K_2(F)$ is canonically a quotient group of A. Here $K_2(F)$ denotes the algebraic K-theory of F (in degree 2).

As F is a non-archimedean local field containing n n-th roots of unity, the Hilbert symbol gives a surjection from $K_2(F)$ to μ_n. Thus in all cases, we have a surjection $A \to \mu_n$, so we can take the quotient of E by the kernel of this surjection to obtain a central extension of G by μ_n, which we shall denote by $\widetilde{G}$. Let p denote the quotient map from $\widetilde{G}$ to G. So we have the exact sequence

$$1 \to \mu_n \to \widetilde{G} \xrightarrow{p} G \to 1.$$

From now on, we shall continue to use the notation $e_\alpha(x)$ and $w_\alpha(x)$ to refer to the corresponding elements of $\widetilde{G}$. These elements naturally satisfy all the relations in Theorem 1.3.1. Also for any subgroup H of G, we shall denote by $\widetilde{H}$ the induced covering group of H.

Define the elements $h_\alpha(x) \in \widetilde{G}$ by $h_\alpha(x) = w_\alpha(x)w_\alpha(-1)$ and let $l_\alpha = ||\alpha^\vee||^2$ where the norm on the coroot lattice has been chosen such that the short coroots have length 1. Then the following identities hold in $\widetilde{G}$ [42, §6].

$$h_\alpha(x)e_\beta(y)h_\alpha(x)^{-1} = e_\beta(x^{\langle\beta,\alpha^\vee\rangle}y) \tag{1.2}$$

$$h_\alpha(x)h_\alpha(y) = (x,y)^{l_\alpha}h_\alpha(xy) \tag{1.3}$$

$$h_\alpha(x)h_\beta(y)h_\alpha(x)^{-1} = h_\beta(y)(x,y)^{\langle\beta,\alpha^\vee\rangle l_\beta} \tag{1.4}$$

$$h_\alpha(x) = h_{-\alpha}(x^{-1}). \tag{1.5}$$

Recall that $(x,y) \in \mu_n$ in the above equations is the value of the Hilbert symbol and is central in $\widetilde{G}$.

1.4 Matsumoto's construction

In this section we present a construction of the central extension $\widetilde{G}$ due to Matsumoto [30]. The necessary datum needed to carry out this construction is a split reductive group $\mathbf{G}$ together with a W-invariant integer valued quadratic form Q on Y. In the case where $\mathbf{G}$ is simple and simply connected, we recover the construction of the previous section by taking Q to be the unique W-invariant quadratic form that takes the value 1 on a short coroot.

In this section, we will work under the hypothesis that $\mu_{2n} \subset F$. We will construct a section $s : G \longrightarrow \widetilde{G}$ and a 2-cocycle σ on G with values in μ_n such that the multiplication in $\widetilde{G}$ is given by $s(g_1)s(g_2) = s(g_1g_2)\sigma(g_1, g_2)$. We shall write elements of $\widetilde{G}$ as ordered pairs (g, ζ) where $\zeta \in \mu_n$, the section s is given by $s(g) = (g, 1)$ and multiplication is given by

$$(g_1, \zeta_1)(g_2, \zeta_2) = (g_1g_2, \zeta_1\zeta_2\sigma(g_1, g_2)).$$

Choosing a basis $e_1, \ldots, e_r$ for Y yields an explicit isomorphism $(F^\times)^r \simeq T$, namely $(t_1, \ldots, t_r) \mapsto t_1^{e_1} \ldots t_r^{e_r}$. Suppose that such an ordered basis has been chosen. For $y = \sum_i y_ie_i$, suppose $Q(y) = \sum_{i \le j} q_{ij}y_iy_j$. Then define a cocycle $\sigma \in H^2(T, \mu_n)$ by

$$\sigma(s,t) = \prod_{i \le j} (s_i, t_j)^{q_{ij}} \tag{1.1}$$

It is straightforward to verify that σ is a 2-cocycle. Thus it defines a central extension $\widetilde{T}$ of T by μ_n. We will now show that this cocycle σ is (cohomologically) W-invariant, this will be necessary to extend our construction of $\widetilde{T}$ to a central extension $\widetilde{M}$ of $M = N_G(T)$.

There is an action of W on Y. Using this action, we define integers $a_{ij}^{(w)}$, for $w \in W$ and $i, j = 1, \ldots, r$ by

$$we_i = \sum_j a_{ij}^{(w)} e_j.$$

The condition that W acts on Y is equivalent to the identity

$$a_{ij}^{(w_1w_2)} = \sum_k a_{ik}^{(w_2)} a_{kj}^{(w_1)}, \tag{1.2}$$

while the condition that $Q(wy) = Q(y)$ implies that

$$\sum_{k\le l} a_{ik}^{(w)} a_{il}^{(w)} q_{kl} = q_{ii} \tag{1.3}$$

and for $i < j$,

$$\sum_{k\le l} (a_{ik}^{(w)} a_{jl}^{(w)} + a_{jk}^{(w)} a_{il}^{(w)}) q_{kl} = q_{ij}. \tag{1.4}$$

We define the integers m_{ij} by

$$m_{ij} = \sum_{k\le l} a_{jk}^{(w)} a_{il}^{(w)} q_{kl}$$

and for all $w \in W$ and $t \in T$, we define

$$\phi_w(t) = \prod_{i<j} (t_i, t_j)^{m_{ij}}.$$

The following identity, which is readily checked to follow from the equations (1.3) and (1.4), shows that σ is cohomologically W-invariant.

$$\frac{\sigma(s,t)}{\sigma(ws,wt)} = \frac{\phi_w(s)\phi_w(t)}{\phi_w(st)}.$$

For each $w \in W$, we now define Φ_w, an automorphism of $\widetilde{T}$ by

$$\Phi_w((t,\zeta)) = (wt, \zeta\phi_w(t)).$$

This definition has the following important property.

Proposition 1.4.1. *The assignment $w \mapsto \Phi_w$ defines an action of W on $\widetilde{T}$ by group automorphisms.*

Proof. This is a straightforward, if slightly tedious, consequence of the identities (1.2), (1.3) and (1.4). □

To each simple coroot α, we have an associated element $w_\alpha \in G$ as defined in Section 1.2. This collection of w_α as α runs over all simple coroots, together with the torus T, generate M. The w_α's satisfy the braid relations, but do not necessarily square to the identity. Let W_0 denote the subgroup of M generated by the w_α. Then W_0 is a finite cover of W, and the intersection of W_0 with T is central in $\widetilde{T}$. Note that it is at this point in the construction where we make use of the fact that $\mu_{2n} \subset F$. We know it is possible (for example using [12]) to discard this assumption, at the cost of more work and greater complication in formulae from this point onwards.

Since W is a quotient of W_0, our action $w \mapsto \Phi_w$ lfts to an action of W_0 on $\widetilde{T}$. Now we can construct the central extension of M as follows: As a set $\widetilde{M} = (\widetilde{T} \times W_0)/\sim$ where the equivalence relation $\sim$ is defined by $(t't, w) \sim (t', tw)$ for $t \in T \cap W_0$. The multiplication in $\widetilde{M}$ is defined by

$$(t_1, w_1)(t_2, w_2) = (t_1 \Phi_{w_1}(t_2), w_1 w_2). \tag{1.5}$$

It is routine to show that this definition of multiplication defines the structure of a group on $\widetilde{M}$. The section $s: T \longrightarrow \widetilde{T}$ is extended to a section $s: M \longrightarrow \widetilde{M}$ by defining $s((t, w)) = (s(t), w)$.

To complete the construction of the local metaplectic group $\widetilde{G}$, we follow the method of Matsumoto [30]. The Bruhat decomposition in G states that $G = UMU$ and that the function $\rho: G \longrightarrow M$ given by $\rho(u_1 m u_2) = m$ for $u_1, u_2 \in U$ and $m \in M$ is well defined. Now define X to be the following set:

$$X = \{(g, m) \in G \times \widetilde{M} \mid \rho(g) = p(m)\}.$$

We shall now define two groups, L_X, a group of left-automorphisms of X and R_X, a group of right-automorphisms of X.

For $t \in \widetilde{T}$, define $\lambda(t)$ and $\lambda^*(t)$ by

$$\lambda(t)(g, m) = (p(t)g, tm),$$

$$(g, m)\lambda^*(t) = (gp(t), mt).$$

For $u \in U$, define $\mu(u)$ and $\mu^*(u)$ by

$$\mu(u)(g, m) = (ug, m),$$

$$(g, m)\mu^*(u) = (gu, m).$$

We next need (to work to) define another automorphism of X for each simple root α. Define

ν_α and ν^*_α by

$$\nu_\alpha(g,m) = (w_\alpha g, s(\rho(w_\alpha g)\rho(g)^{-1})m),$$

$$(g,m)\nu^*_\alpha = (gw_\alpha, m.s(\rho(gw_\alpha)^{-1}\rho(g))^{-1}).$$

It is straightforward to check that each of the automorphisms defined above do map X to X.

We let L_X be the group generated by all $\lambda(h)$, $\mu(u)$ and ν_α, and let R_X be the group generated by all $\lambda^*(h)$, $\mu^*(u)$ and ν^*_α.

Lemma 1.4.2. *The left action of L_X commutes with the right action of R_X, that is $g(xg') = (gx)g'$ for all $g \in L_X$, $g' \in R_X$ and $x \in X$.*

Proof. It suffices to check this identity for all choices of g and g' running through generating sets of L_X and R_X respectively. For most such choices of g and g' this is striaightforward, and it is only the case where $g = \nu_\alpha$ and $g' = \nu^*_\beta$ that requires more work. To achieve this requires a lemma about how the function ρ behaves with respect to left and right multiplication by w_α.

Lemma 1.4.3. *Suppose $g = u_1 e_\alpha(x) m e_\alpha(y) u_2$ with $u_1, u_2 \in U_\alpha$, $x, y \in F$ and $m \in M$. Write w for the image of m in W. Then we have*

$$\rho(w_\alpha g) = \begin{cases} w_\alpha \rho(g) & \textit{if } w^{-1}\alpha > 0 \textit{ or } x = 0 \\ h_\alpha(\frac{1}{x})\rho(g) & \textit{if } w^{-1}\alpha < 0 \textit{ and } x \neq 0 \end{cases}$$

and

$$\rho(g w_\alpha) = \begin{cases} \rho(g) w_\alpha & \textit{if } w\alpha > 0 \textit{ or } y = 0 \\ \rho(g) h_\alpha(y) & \textit{if } w\alpha < 0 \textit{ and } y \neq 0. \end{cases}$$

Proof. These particular formulae follow from the structure theory of split reductive groups over a field. In particular, a reader acquainted with the identities presented in [42] may easily derive them. □

Using this lemma, the proof of the identity $(\nu_\alpha x)\nu^*_\beta = \nu_\alpha(x\nu^*_\beta)$ is easily completed except in the cases where $\alpha = \pm w\beta$, which we shall now enter into in greater detail.

Restricting to this particular case reduces the problem to a calculation in a rank one group. We see from Lemma 1.4.3 that there will be a number of cases to consider according to whether or not certain variables are zero. In this text, we shall only deal with the generic (in the Zariski sense of the word) case. This is the most involved case, and the only time in the construction of $\widetilde{G}$ that the identity $(z, 1-z) = 1$ is used.

Suppose g is an element of a rank one group, we do not yet make a distinction between SL_2 and PGL_2. In the generic case, g lies in the big Bruhat cell, so can be written.

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & u \\ t & 0 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}.$$

The following matrix identity is fundamental

$$\begin{pmatrix} 0 & -1 \\ l & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & u \\ t & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1/x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & u/x \\ tx & 0 \end{pmatrix} \begin{pmatrix} 1 & u/(tx) \\ 0 & l \end{pmatrix}.$$

Let $x' = x + u/(ty)$ and $y' = y + u/(tx)$. In verifying that $(\nu_\alpha x)\nu_\alpha^* = \nu_\alpha(x\nu_\alpha^*)$, the first coordinates are clearly equal, so we only need to consider the second coordinate. Write $m = \rho(g)$. The above equations reduce this to checking that

$$s(h_\alpha(x^{-1}))ms(h_\alpha(y')) = s(h_\alpha(x'^{-1}))ms(h_\alpha(y)).$$

Let us first consider the case of SL_2. Our knowledge of the multiplication in $\widetilde{M}$ from (1.5) and the section s restricted to M allows us to reduce this identity to

$$(\frac{1}{x}, u)(\frac{u}{x}, \frac{1}{y'}) = (\frac{1}{x'}, u)(\frac{u}{x'}, \frac{1}{y}).$$

Writing $x'/x = y'/y = z$ and using the multiplicativity properties of the Hilbert symbol, this reduces to $(\frac{u^2}{xy}, z) = 1$, which is true since $1 - z = \frac{u^2}{xy}$.

In the PGL_2 case, we proceed in the same manner as for the SL_2 case. This time we may assume without loss of generality that $t = 1$. We find that the corresponding identity that is needed to be proved is

$$(\frac{1}{x^2}, u)(\frac{u}{x^2}, \frac{1}{y'^2}) = (\frac{1}{x'^2}, u)(\frac{u}{x'^2}, \frac{1}{y^2})$$

and the technique of proof is exactly the same.

It was remarked earlier that the other cases where g is not in the Bruhat cell, or when

x, y, x' or y' are zero need to be dealt with individually. This can be completed on a case by case basis which we do not bother with here, as it is routine and considerably simpler than the case dealt with here. $\square$

Corollary 1.4.4. *The group L_X acts on X in a simply transitive manner*

Proof. Since G is generated by T, U and all w_α, for any $(g, m) \in X$ we can find g_1 such that $g_1(g, m) = (1, \zeta)$. $\zeta \in \mu_n$ by the condition $(1, \zeta) \in X$. Then apply the element $\lambda(\zeta)$ for $\zeta \in \mu_n \subset \widetilde{T}$ to show that (g, m) is in the same L_X-orbit of $(1, 1)$. Hence the action of L_X on X is transitive, and similarly the action of R_X on X is transitive. To show that the action is simple, suppose that $gx = x$. Then for all $g' \in G$ we have $gxg' = xg'$. By transitivity of the R_X-action, we have $gy = y$ for all $y \in X$. Since L_X is a subgroup of the automorphism group of X, this implies that $g = 1$, so the action is simply transitive. $\square$

The group L_X is our desired central extension $\widetilde{G}$ of G that we have set out to construct. The projection map $p\colon L_X \longrightarrow G$ is induced by the projection map $X \to G$ (with G acting on itself by left multiplication). The kernel of p is μ_n, since any such element must map $(1, 1)$ to some $(1, \zeta)$. We also have an isomorphism $p^{-1}(M) \simeq \widetilde{M}$.

If we restrict ourselves to the case of SL_2, then we can be very explicit about the resulting 2-cocycle. Following Kubota [24], we have the following formula for $\sigma \in H^2(SL_2(F), \mu_n)$.

$$\sigma(g, h) = \left(\frac{x(gh)}{x(g)}, \frac{x(gh)}{x(h)} \right)^{Q(\alpha)}, \tag{1.6}$$

where for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(F)$, we define $x(g) = c$ unless $c = 0$ in which case $x(g) = d$. In this formula, α is a simple coroot.

1.5 General existence of the central extension

Now we turn to the more general situation where we relax our datum of Y to that of a W-invariant integer valued bilinear form B on Y for which $B(\alpha^\vee, \alpha^\vee) \in 2\mathbb{Z}$ for all coroots $\alpha^\vee$. Building on the work of several authors, our major immediate goal is the proof of Theorem 1.5.1, which gives the existence of the family of central extensions which we study.

Theorem 1.5.1. *With our assumptions on G, F, n and B as above, there exists a central extension $\widetilde{G}$ of G by μ_n, such that when restricted to a central extension of T, the* *commutator* $[\cdot,\cdot]: T \times T \longrightarrow \mu_n$ *is given by*

$$[x^\lambda, y^\mu] = (x,y)^{B(\lambda,\mu)}$$

for all $x, y \in F^\times$ *and* $\lambda, \mu \in Y$.

A more explicit form of the commutator formula is possible. Pick a basis $e_1, \dots, e_r$ of Y. This induces an explicit isomorphism $(F^\times)^r \simeq T$, namely $(t_1, \dots, t_r) \mapsto t_1^{e_1} \dots t_r^{e_r}$. Via this isomorphism, suppose that $s = (s_1, \dots, s_r)$ and $t = (t_1, \dots, t_r)$ are two elements of T. Then their commutator is given by the formula

$$[s,t] = \prod_{i,j} (s_i, t_j)^{b_{ij}} = \prod_i (s_i, \prod_j t_j^{b_{ij}}) \tag{1.1}$$

where the integers b_{ij} are defined in terms of the bilinear form B by

$$B(\sum_i x_i e_i, \sum_j y_j e_j) = \sum_{i,j} b_{ij} x_i y_j.$$

We first show how to deduce Theorem 1.5.1 from the semisimple simply connected case and the torus case. This argument follows the proof of [18, Proposition 1]. For our purposes here, it will be most convenient for us to reinterpret a central extension of a group H by a group A as a morphism of group stacks $H \to BA$, where BA is the stack classifying A-bundles.

We briefly indicate how this translation works. By definition, a morphism $H \to BA$ is an A-torsor over H. Since A is abelian, the multiplication map from $A \times A$ to A is a group homomorphism and this induces a group structure on BA. The extra structure of a group homomorphism on the map $H \to BA$ is what yields the datum of a group structure on the total space of the torsor represented by this map. It is this group structure on this total space which is the central extension of H by A.

Let G^{sc} be the simply connected cover of the derived group of G, and let T^{sc} be the inverse image of T in G^{sc}. Suppose that we have two central extensions $G^{sc} \to B\mu_n$ and $T \to B\mu_n$ which are isomorphic upon restriction to T^{sc}. Suppose furthermore that the extension $G^{sc} \to B\mu_n$ is invariant under the conjugation action of T on G^{sc} and the trivial T-action on μ_n. To this set of data, we construct a central extension $G \to B\mu_n$.

Consider the semidirect product $G^{sc} \rtimes T$ where T is acting on G^{sc} by conjugation. The datum we have of group morphisms $T \to B\mu_n$ and $G^{sc} \to B\mu_n$ with the latter being T-equivariant is equivalent to that of a group morphism $G^{sc} \rtimes T \to B\mu_n$.

Now consider the multiplication map from $G^{sc} \rtimes T$ to G. This is a surjective group homomorphism with kernel isomorphic to T^{sc} embedded inside $G^{sc} \rtimes T$ via $t \mapsto (t, t^{-1})$.

Since the restrictions of $G^{sc} \to B\mu_n$ and $T \to B\mu_n$ to T^{sc} are assumed isomorphic, the restriction of $G^{sc} \rtimes T \to B\mu_n$ is a trivial central extension. Choosing a trivialisation, we may thus factor $G^{sc} \rtimes T \to B\mu_n$ through the quotient G of $G^{sc} \rtimes T$ by T^{sc} and thus we get our desired extension $G \to B\mu_n$.

We now discuss the simply connected semisimple case. In this case, the existence of our desired central extensions of G by μ_n are well-known to exist. Steinberg [41] computed the universal central extension of G and Matsumoto [30] computed the kernel of this extension, showing it to be equal to the group $K_2(F)$ except when G is of symplectic type, in which case $K_2(F)$ is canonically a quotient of this kernel. In fact, the adjoint group always acts by conjugation on the universal central extension, and the group of coinvariants is always $K_2(F)$. More recently, Brylinski and Deligne [12] have generalised this construction, proving the following.

Proposition 1.5.2. *The category of central extensions of* $\mathbf{G^{sc}}$ *by* $\mathbf{K}_2$ *as sheaves on the big Zariski site* $Spec(F)_{Zar}$ *is equivalent to the category of integer valued Weyl group invariant quadratic forms on* Y^{sc}*, where the only morphisms in the latter category are the identity morphisms.*

This reproduces the classical construction upon taking F-points. It is most important to us that this central extension is both derived from a solution to a universal problem (so that any automorphism of G^{sc} lifts to an automorphism of the extension), and that this extension has no automorphisms.

By our assumption that $B(\alpha, \alpha) \in 2\mathbb{Z}$ for all coroots α, there exists a $\mathbb{Z}$-valued quadratic form Q on Y^{sc} whose associated bilinear form $Q(x+y) - Q(x) - Q(y)$ is B. We consider the central extension of Proposition 1.5.2. At the level of F-points, this yields a central extension

$$1 \to K_2(F) \to E \to G \to 1.$$

Since F is assumed to be a local non-archimedean field containing n n-th roots of unity, there is a surjection $K_2(F) \to \mu_n$ given by the Hilbert symbol and hence we obtain our central extension

$$1 \to \mu_n \to \widetilde{G} \to G \to 1.$$

This is known to have the desired commutator relation [12, Proposition 3.14] upon restriction to the torus.

In particular, it is important to note that with T acting on G^{sc} by conjugation, since the extension by K_2 is a canonically constructed object that has no automorphisms, the action of T extends to an action on the cover $\widetilde{G}^{sc}$ that is trivial when restricted to the central μ_n.

In the torus case, the construction of [12] does not produce all natural central extensions. For example, even in the case of GL_1 over a field of Laurent series, the work of [2] produces a central extension whose commutator is the Hilbert symbol, whereas K_2-based methods only produce the square of this extension. The author does not know how to generalise this construction to the case of mixed characteristic, so we shall proceed in the following ad hoc manner, making use of the assumption that $\mu_{2n} \subset F$. We will write $(\cdot,\cdot)_{2n}$ for the Hilbert symbol with values in μ_{2n} and reserve $(\cdot,\cdot)$ for the Hilbert symbol with values in μ_n.

We can associate to $2B$ a quadratic form Q and consider the corresponding central extension

$$1 \to \mu_{2n} \to E \to T \to 1.$$

This has commutator

$$[x^\lambda, y^\mu] = (x,y)_{2n}^{2B(\lambda,\mu)} = (x,y)^{B(\lambda,\mu)}.$$

We will realise $\widetilde{T}$ as an index 2 subgroup of E.

Since the commutator takes values only in the subgroup μ_n of μ_{2n}, when we quotient out by μ_n, we obtain a central extension

$$1 \to \mu_2 \to E' \to T \to 1.$$

where the group E' is abelian.

The central extension splits over $\mathbf{T}(O_F)$. Since the Hilbert symbol is trivial on $O_F^\times \times O_F^\times$, this is most obvious when we express the central extension in terms of a 2-cocycle; one such choice is

$$\sigma(s,t) = \prod_{i\leq j} (s_i, t_j)_{2n}^{q_{ij}}$$

where $Q(\sum_i y_i e_i) = \sum_{i\leq j} q_{ij} y_i y_j$.

Now choose a splitting of $\mathbf{T}(O_F)$ and quotient E' by its image. We arrive at a central extension

$$1 \to \mu_n \to E'' \to T/\mathbf{T}(O_F) \to 1.$$

This is a short exact sequence of abelian groups with the last term free. Hence it splits, so we get a surjection $E \to \mu_2$. Taking the kernel of this surjection as $\widetilde{T}$, we get our desired central extension of T by μ_n.

It remains to show that the two extensions of T and G^{sc} associated to B agree upon restriction to T^{sc}. This may be seen by an analogous argument to the one above: both extensions split over $\mathbf{T}^{\mathbf{sc}}(O_F)$ and have the same commutator, so their difference in $H^2(T, \mu_n)$ is abelian, splitting over $\mathbf{T}^{\mathbf{sc}}(O_F)$, hence trivial.

1.6 Splitting Properties

A subgroup H of G is said to be split by the central extension if we have an isomorphism of groups $p^{-1}(H) \simeq \mu_n \times H$ that commutes with the projection maps to H. In this section, we shall show that the unipotent subgroups, and the maximal compact subgroup K are split in $\widetilde{G}$. We also discuss splittings of discrete subgroups corresponding to the coroot lattice and the Weyl group.

Proposition 1.6.1. *Any unipotent subgroup U of G is split canonically by the central extension $\widetilde{G}$.*

Proof. This result is proved in greater generality in [34, Appendix 1]. Since we are concerned only with the case where $(n, q) = 1$, a simple proof can be given. The assumptions on n ensure that the map $U \to U$ given by $u \mapsto u^n$ is bijective. If $u \in U$, write $u = u_1^n$ and let $\tilde{u_1}$ be any lift of u_1 to $\widetilde{G}$. Then define $s(u) = (\tilde{u_1})^n$. This is well defined, invariant under conjugation and is the proposed section determining the splitting.

So it suffices to show that s is a group homomorphism. For U abelian this is trivial. In general, U is solvable, write U' for the quotient group $U/[U,U]$ and by induction, we may assume that s is a homomorphism when restricted to $[U,U]$. We now form the quotient $\widetilde{U}' = \widetilde{U}/s([U,U])$. Suppose $u_1, u_2 \in U$. Form $\zeta = s(u_1)s(u_2)s(u_1u_2)^{-1}$. A priori, we have $\zeta \in \mu_n$. Projecting into $\widetilde{U}'$ and using the abelian case of the proposition implies that $\zeta \in s([U,U])$ and thus we have $\zeta = 1$ so s is a homomorphism as desired. □

For the corresponding result for the maximal compact subgroup, the splitting is no longer unique, and in order for the splitting to exist, it is essential that n is coprime to the residue characteristic. In the simply connected case, this is an old result of Moore [35, Lemma 11.3].

Theorem 1.6.2. *[12, §10.7] The extension $\widetilde{G}$ of G splits over the maximal compact subgroup $K = \mathbf{G}(O_F)$.*

Proof. Let K_1 denote the kernel of the surjection $\mathbf{G}(O_F) \to \mathbf{G}(k)$. Then K_1 is a pro-p group, hence has trivial cohomology with coefficients in μ_n. By the Lyndon-Hochschild-Serre spectral sequence, we thus have an isomorphism $H^2(K, \mu_n) \simeq H^2(\mathbf{G}(k), \mu_n)$. Since the index of $\mathbf{M}(k)$ in $\mathbf{G}(k)$ is coprime to n, we know that the map $H^2(\mathbf{G}(k), \mu_n) \to H^2(\mathbf{M}(k), \mu_n)$ is injective. $\mathbf{T}(k)$ can be considered as the k points of the group scheme of $(q-1)$-th roots of unity in $\mathbf{T}$, which is etale, so lifts uniquely into $\mathbf{T}(O_F)$. The group generated by this lift together with the elements $w_\alpha \in K$ form a lift of $\mathbf{M}(k)$ into K. Thus it suffices to show that σ, the defining 2-cocyle for $\widetilde{G}$, is trivial when restricted to this lift of $\mathbf{M}(k)$ in K. But we have an explicit formula for the multiplication in this subgroup. Since all such roots of unity have valuation 1 and the Hilbert symbol is trivial when restricted to $O_F^\times$, we see that σ is trivial on the lift of $\mathbf{T}(k)$. In the explicit construction of $\widetilde{M}$ in Section 1.5, the action of W on $p^{-1}(\mathbf{T}(k))$ commutes with the trivial splitting isomorphism $p^{-1}(\mathbf{T}(k)) \simeq \mathbf{T}(k) \times \mu_n$, which shows that the lift of $\mathbf{M}(k)$ splits, as desired. □

When needed, we will denote by $\kappa^*(k) = s(k)\kappa(k)$ the lifting of K. $(\kappa : K \longrightarrow \mu_n)$. This map κ is not unique, being well defined only up only to a homomorphism from K to μ_n. In [22] a canonical choice is made in the case of $G = GL_n$, however this failure of uniqueness shall not be of concern to us.

Just as in the case of $G = SL_2$ when we were able to provide an explicit formula for the cocycle σ, again in this case we are able to provide an explicit formula for the splitting κ, following Kubota [24]. We have

$$\kappa \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} (c,d)^{Q(\alpha)} & \text{if } 0 < |c| < 1 \\ 1 & \text{otherwise,} \end{cases} \tag{1.1}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(O_F)$, where again α is the simple coroot.

The group Y, considered as a subgroup of T by the injection $\lambda \mapsto \varpi^\lambda$ is trivially split by the section s. The cover W_0 of the Weyl group W is split by s by construction. Let $W_{a,0}$ be the corresponding cover of the affine Weyl group, that is the group generated by Y and W_0. Since this group is a product of its generating subgroups, we see by following through the construction of $\widetilde{M}$ that this subgroup is also split by the section s. For any $w_a \in W_{a,0}$, we will identify w_a with its image under s in $\widetilde{G}$.

Chapter 2

Representation Theory

2.1 Heisenberg group representations

A Heisenberg group is defined to be any two-step nilpotent subgroup, i.e. H is a Heisenberg group if its commutator subgroup is abelian. The metaplectic torus $\widetilde{T}$ is an example of such a group, since the commutator subgroup $[\widetilde{T}, \widetilde{T}]$ is contained in the central μ_n. The representation theory of Heisenberg groups is well understood, in particular we will make use of the following version of the Stone-von Neumann theorem, compare for example with [43, Theorem 3.1].

Theorem 2.1.1. *Let H be a Heisenberg group with centre Z such that H/Z is finite and let χ be a character of Z. Suppose that $\ker(\chi) \cap [H, H] = \{1\}$. Then up to isomorphism, there is a unique irreducible representation of H with central character χ. It can be constructed as follows: Let A be a maximal abelian subgroup of H and let χ' be any extension of χ to A. Then inducing this representation from A to H produces the desired representation.*

Proof. Let π be an irreducible H-representation with central character χ. Since H/Z is finite, π is finite dimensional. Considering π as an A-representation, this implies that it has a one-dimensional quotient χ' which must be an extension of the character χ to A.

By Frobenius reciprocity, there is thus a non-trivial H-morphism from π to $\operatorname{Ind}_A^H \chi'$. To conclude that π is isomorphic to $\operatorname{Ind}_A^H \chi'$, we need to prove that this induced representation is irreducible.

Since $\operatorname{Ind}_A^H \chi'$ is generated by a non-zero function supported on A, to prove irreducibility, it suffices to show that any H-invariant subspace contains such a function. So suppose $f \neq 0$

is in a H-invariant subspace M. Then by translating by an element of H, we may assume that the support of f contains A. Now suppose that there exists $h \in H \setminus A$ such that $f(h) \neq 0$. Then since A is a maximal abelian subgroup of H, there exists $a \in A$ such that $[h, a] \neq 1$. Now consider $(a - \epsilon([a, h])\chi'(a))f \in M$. It has strictly smaller support than f (since it vanishes at h) and is non-zero on A. So continual application of this method will, by finite dimensionality, produce a non-zero function in M supported on A and thus we've proved the irreducibility of $\mathrm{Ind}_A^H \chi'$.

To finish the proof of the theorem, we need to show that if we have two different extensions χ_1 and χ_2 of χ to A, then after inducing to H, we get isomorphic representations. Given two such extensions, $\chi_1\chi_2^{-1}$ is a character of A/Z, and we extend it to a character of H/Z. Our assumption that $\ker(\chi) \cap [H, H] = \{1\}$ implies that the pairing $\langle \cdot, \cdot \rangle : H/Z \times H/Z \longrightarrow \mathbb{C}^\times$ given by $\langle h_1, h_2 \rangle = \chi([h_1, h_2])$ is nondegenerate and hence that every character of H/Z is of the form $h \mapsto \chi([h, x])$ for some $x \in H$. Hence there exists $x \in H$ such that $\chi_1\chi_2^{-1}(a) = \chi([a, x])$ for all $a \in A$. This implies that the characters χ_1 and χ_2 conjugate by x under the conjugation action of H on A. Hence when induced to representations of H, the induced representations are isomorphic. ☐

Corollary 2.1.2. *Genuine representations of $\widetilde{T}$ are parametrised by characters of $Z(\widetilde{T})$.*

Proof. $\widetilde{T}$ is a Heisenberg group so we only need to check that the conditions of the above theorem are satisfied. The condition that $\widetilde{T}/Z(\widetilde{T})$ is finite is satisfied since T^n (the subgroup of n-th powers in T) is central and $(F^\times)^n$ is of finite index in $F^\times$. The condition that $[H, H] \cap \ker(\chi) = \{1\}$ is satisfied for genuine characters χ, since $[\widetilde{T}, \widetilde{T}] \subset \mu_n$ and χ is faithful on μ_n. Hence we may apply Theorem 2.1.1 to obtain our desired corollary. ☐

We now produce explicitly a choice of maximal abelian subgroup of $\widetilde{T}$ that we can use later on for our convenience.

Lemma 2.1.3. *The group $Z_{\widetilde{T}}(\widetilde{T} \cap K)$ is a maximal abelian subgroup of $\widetilde{T}$.*

Proof. Since $\widetilde{T} \cap K$ is abelian, it is clear that any maximal abelian subgroup of $\widetilde{T}$ containing $\widetilde{T} \cap K$ is contained in $Z_{\widetilde{T}}(\widetilde{T} \cap K)$. So it suffices to prove that $Z_{\widetilde{T}}(\widetilde{T} \cap K)$ is abelian.

Recall the basis $e_1, \ldots, e_r$ of Y which was used to introduce coordinates on T. Define integers b_{ij} by $B(x, y) = \sum_{i,j} b_{ij} x_i y_j$ where $x = \sum_i x_i e_i$, $y = \sum_i y_i e_i$ and $B(x, y) = Q(x + y) - Q(x) - Q(y)$ is the symmetric bilinear form corresponding to Q. Then the commutator of two elements in $\widetilde{T}$ is given by the formula (1.1) which we repeat here. (see also [12, Corollary 3.14])

$$[s,t] = \prod_{i,j}(s_i,t_j)^{b_{ij}} = \prod_i (s_i, \prod_j t_j^{b_{ij}}).$$

Thus the condition for t to be in $Z_{\widetilde{T}}(\widetilde{T} \cap K)$ is that $\prod_j t_j^{b_{ij}}$ has valuation divisible by n for all i. Now suppose that s and t are elements of $Z_{\widetilde{T}}(\widetilde{T} \cap K)$. Let x_i and y_i be the valuations of s_i and t_i respectively. Then we have that (s_i,t_j) is equal to $((-1)^{x_i y_j} s_i^{y_j}/t_j^{x_i})^{\frac{q-1}{n}}$ after reduction modulo ϖ. Hence we may compute the commutator

$$[s,t] = \left(\prod_{i,j}(-1)^{x_i y_j} \prod_i s_i^{\sum_j y_j b_{ij}} \prod_j t_j^{\sum_i x_i b_{ij}}\right)^{\frac{q-1}{n}}.$$

Since we are assuming that $2n|q-1$, the power of -1 which appears in this product is even. By assumption on s and t, all exponents of s_i and t_j are divisible by n. So the whole product is a $q-1$-th power, so after reduction modulo ϖ becomes 1. Hence s and t commute, so $Z_{\widetilde{T}}(\widetilde{T} \cap K)$ is abelian, as required. □

We will use H to denote this maximal abelian subgroup $Z_{\widetilde{T}}(\widetilde{T} \cap K)$.

2.2 Principal Series Representations

We begin by studying the class of representations that will be our main object of study. Let (π, V) be a genuine, smooth admissible representation of $\widetilde{T}$. The group $\widetilde{B}$ contains U as a normal subgroup with quotient naturally isomorphic to $\widetilde{T}$. Via this quotient, we consider (π, V) as a representation of $\widetilde{B}$ on which U acts trivially.

Definition 2.2.1. *For (π, V) a smooth representation of $\widetilde{T}$, we define the (normalised) induced representation $I(V)$ as follows:*

The space of $I(V)$ is the space of all locally constant functions $f : \widetilde{G} \longrightarrow V$ such that

$$f(bg) = \delta^{1/2}(b)\pi(b)f(g)$$

for all $b \in \widetilde{B}$ and $g \in \widetilde{G}$ where δ is the modular quasicharacter of $\widetilde{B}$ and we are considering (π, V) as a representation of $\widetilde{B}$. The action of $\widetilde{G}$ on $I(V)$ is given by right translation. In *this way we define an induction functor*

$$I\colon Rep\,(\widetilde{T}) \longrightarrow Rep\,(\widetilde{G}).$$

Suppose now that χ is a genuine character of $Z(\widetilde{T})$. We denote by $i(\chi) = (\pi_\chi, V_\chi)$ a representative of the corresponding isomorphism class of irreducible representations of $\widetilde{T}$ with central character χ. By the considerations in the above section, $i(\chi)$ is finite dimensional. We will write $I(\chi)$ for the corresponding induced representation $I(i(\chi))$ of $\widetilde{G}$. Such representations $I(\chi)$ will be called *principal series representations.*

We now define a family of principal series representations, called *unramified* that will be of principal interest to us.

Definition 2.2.2. *A genuine character χ of $Z(\widetilde{T})$ is said to be unramified if it has an extension to H that is trivial on $\widetilde{T} \cap K$. We use the same adjective unramified for the corresponding representation $I(\chi)$ of $\widetilde{G}$.*

The following important Lemma is largely contained in the works of Kazhdan and Patterson [22] and Savin [39]. This work contains the added generality necessary to state this result precisely as required for our applications.

Lemma 2.2.3. *An unramified principal series representation $I(\chi)$ has a one-dimensional space of K-fixed vectors.*

Proof. Suppose $f \in I(\chi)^K$. Let $g = f(1) \in i(\chi)$. By the Iwasawa decomposition $G = UAK$, we may write any $g \in \widetilde{G}$ as $g = uak$ with $u \in U$, $t \in \widetilde{T}$ and $k \in K$. Then we have $f(g) = f(uak) = \pi_\chi(a)g$.

The element a is well defined up to right multiplication by en element $\eta \in \widetilde{T} \cap K$. This induces (the only) compatibility condition, we thus require that $f(g) = \pi_\chi(a\eta)g = \pi_\chi(a)\pi_\chi(\eta)g$. Thus we have that $f \mapsto f(1)$ is an isomorphism from $I(\chi)^K$ to $i(\chi)^{\widetilde{T}\cap K}$.

If $g \in i(\chi)^{\widetilde{T}\cap K}$, then for all $t \in \widetilde{T}$ and $\eta \in \widetilde{T} \cap K$ we have

$$g(t) = g(t\eta) = [t,\eta]g(\eta t) = [t,\eta]\chi(\eta)g(t).$$

Since χ is unramified, $\chi(\eta) = 1$, so either $[t,\eta] = 1$ or $g(t) = 0$. The function $g \in i(\chi)$ is determined by its restriction to a set of coset representatives of $H\backslash\widetilde{T}$. By the definition of

H we know that $[t,\eta]=1$ for all $\eta \in \widetilde{T} \cap K$ if and only if $t \in H$. Thus we have shown that $i(\chi)^{\widetilde{T}\cap K}$ is one dimensional, proving the lemma. $\square$

A K-fixed vector in such a representation will be called *spherical.*

We now define an action of the Weyl group W on principal series representations.

The group $\widetilde{M}$ acts on $\widetilde{T}$ by conjugation, and hence acts on $\mathrm{Rep}\,(\widetilde{T})$. Explicitly, write $c_m : \widetilde{T} \longrightarrow \widetilde{T}$ for the operation of conjugation by $m \in \widetilde{M}$ on $\widetilde{T}$. Then for (π, V), the action of $m \in \widetilde{M}$ is defined by $(\pi, V)^m = (\pi^m, V^m)$ where $V^m = V$ and π^m is the composition $\pi \circ c_m : \widetilde{T} \longrightarrow \mathrm{Aut}\,(V)$.

Unfortunately when we restrict this action to $\widetilde{T}$, we do not obtain the identity. However we may still define an action of the Weyl group on $\mathrm{Rep}\,(\widetilde{T})$. Recall from the discussion at the end of section 1.6, that the group W_0 lifts to $\widetilde{M}$. In this reaslisaton of W_0, the kernel of the surjection $W_0 \to W$ lies in $Z(\widetilde{T})$. Since the conjugation action of $Z(\widetilde{T})$ is trivial, we are able to define an action of W on $\mathrm{Rep}\,(\widetilde{T})$ by first restricting the action of $\widetilde{M}$ to an action of W_0, which then induces a well-defined action of W on $\mathrm{Rep}\,(\widetilde{T})$.

In a similar but simpler manner, one may define an action of W on the space of characters of $Z(\widetilde{T})$. These two actions are compatible in the sense that $i(\chi^w) = i(\chi)^w$ for all characters χ and $w \in W$.

To proceed, we require the theory of the Jacquet functor, and the results regarding the composition of the Jacquet functor with the induction functor I. These results are all contained in [5].

Definition 2.2.4. *The Jacquet functor J from $Rep(\widetilde{G})$ to $Rep(\widetilde{T})$ is defined to be the functor of U-coinvariants.*

Explicitly, for an object V in $\mathrm{Rep}\,(\widetilde{G})$, $J(V)$ is defined to be the largest quotient of V on which U acts trivially, that is we quotient out by the submodule generated by all elements of the form $\pi(u)v - v$ with $u \in U$ and $v \in V$. Since $\widetilde{T}$ normalises U (as U has a unique splitting, conjugation by $\widetilde{T}$ must preserve this splitting), the action of $\widetilde{T}$ on V induces an action on $J(V)$, so the image of J is indeed in $\mathrm{Rep}\,(\widetilde{T})$.

The work in [5] is in sufficient generality to cover our circumstances. In particular, we have the following two propositions.

Proposition 2.2.5. *[5, Proposition 1.9(a)] The Jacquet functor is exact.*

Proposition 2.2.6. *[5, Proposition 1.9(b)] The Jacquet functor J is left adjoint to the induction functor I. That is, there exists a natural isomorphism*

$$Hom_{\widetilde{T}}(J(V), W) \cong Hom_{\widetilde{G}}(V, I(W)).$$

The main result of [5] is their Theorem 5.2, from which we derive the following important corollary.

Corollary 2.2.7. *The composition factors of the Jacquet module $J(I(\chi))$ are given by $i(\chi^w)$ as w runs over W.*

Proof. The derivation of this corollary follows in exactly the same manner as in the reductive case. In the notation of Bernstein and Zelevinsky [5], we apply their Theorem 5.2 with $\mathbf{G} = \widetilde{G}$, $\mathbf{Q} = \mathbf{P} = \widetilde{B}$, $\mathbf{N} = \mathbf{M} = \widetilde{T}$, $\mathbf{V} = \mathbf{U} = U$ and $\psi = \theta = 1$. □

We say that a character χ of $Z(\widetilde{T})$ is regular if $\chi^w \neq \chi$ for all $w \neq 1$.

Proposition 2.2.8. *If χ is regular, then $J(I(\chi))$ is semisimple.*

Proof. Decompose the $\widetilde{T}$-module $J(I(\chi))$ into $Z(\widetilde{T})$-eigenspaces – this must be a semisimple decomposition since we are dealing with commuting operators on a finite dimensional space. As χ is assumed to be regular, these eigenvalues of $Z(\widetilde{T})$ are all distinct. This shows that the filtration from Corollary 2.2.7 splits as $\widetilde{T}$-modules, so we're done. □

2.3 Intertwining operators

We start with these results on the spaces of morphisms between various principal series representations.

Theorem 2.3.1. *1. For two characters χ_1 and χ_2 of $Z(\widetilde{T})$, we have*

$$Hom_{\widetilde{G}}(I(\chi_1), I(\chi_2)) = 0$$

unless there exists $w \in W$ such that $\chi_2 = \chi_1^w$.

2. Suppose that χ is regular. Then for all $w \in W$ we have

$$\dim\ Hom_{\widetilde{G}}(I(\chi), I(\chi^w)) = 1$$

Proof. Since J is left adjoint to I, we have

$$\mathrm{Hom}_{\widetilde{G}}(I(\chi_1), I(\chi_2)) = \mathrm{Hom}_{\widetilde{T}}(J(I(\chi_1)), i(\chi_2)).$$

Our knowledge of the description of the composition series of $J(I(\chi_1))$ from Corollary 2.2.7 and Proposition 2.2.8 completes the proof. $\square$

This section will be dedicated to the explicit construction and analysis of these spaces $\mathrm{Hom}_{\widetilde{G}}(I(\chi), I(\chi^w))$. Elements in these spaces are referred to as intertwining operators.

Suppose $s \in \mathbb{C}$. Associated to s is a one dimensional representation δ^s of $\widetilde{B}$ given by raising the modular quasicharacter δ to the s-th power. Accordingly, given any representation V of $\widetilde{T}$, we define a family I_s of representations of $\widetilde{G}$ by

$$I_s(V) = I(V \otimes \delta^s).$$

For each $V \in \mathrm{Rep}\,(\widetilde{T})$, this family of representations is a trivialisable vector bundle over $\mathbb{C}$. We choose a trivialisation as follows.

To each $f \in I(V) = I_0(V)$ and $s \in C$, we define the element $f_s \in I_s(V)$ by

$$f_s(bk) = \delta(b)^s f(bk) \tag{2.1}$$

for any $b \in \widetilde{B}$ and $k \in K$. It is easily checked that this is well defined, the claim $f_s \in I_s(V)$ is true and that $s \mapsto f_s$ does define a section.

Our strategy for constructing intertwining operators is as follows. We shall first construct intertwining operators via an integral representation that is only absolutely convergent on a cone in the set of all possible characters χ. We then make use of the trivialising section we have just constructed to meromorphically continue these intertwining operators to all $I(\chi)$.

For any finite dimensional $\widetilde{T}$ representation (π, V), and any coroot α, we define the α-radius $r_\alpha(V)$ to be the maximum absolute value of an eigenvalue of the operator $\pi(\varpi^\alpha)$ on V. This turns out to be independent of the choice of uniformiser ϖ since $T(O_F)$ is compact.

For $w \in W$ and such a finite dimensional representation (π, V), the intertwining operator

$T_w : I(V) \longrightarrow I(V^w)$ is defined by the integral

$$(T_w f)(g) = \int_{U_w} f(w^{-1}ug)du. \tag{2.2}$$

whenever this is absolutely convergent.

To check that T_w does indeed map $I(V)$ into $I(V^w)$ is a simple calculation. Note that the underlying vector spaces of V and V^w are equal as per the definition of the W-action on such representations in section 2.2.

Lemma 2.3.2. *Suppose that $w_1, w_2 \in W$ are such that $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$. Then $T_{w_1w_2} = T_{w_1}T_{w_2}$, whenever their defining integrals are absolutely convergent.*

Proof. This result is a simple application of Fubini's theorem. $\square$

Let us now restrict ourselves to a study of the case where $w = w_\alpha$ is the simple reflection corresponding to the simple coroot α.

Theorem 2.3.3. *The defining integral (2.2) for the intertwining operator T_{w_α} is absolutely convergent for $r_\alpha(V) < 1$.*

Proof. In SL_2, we have the following identity

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/x & 1 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/x & 1 \end{pmatrix}$$

We apply the morphism ϕ_α to interpret this as an identity in G. This equation lifts to $\widetilde{G}$ as the relevant Kubota cocycles are trivial. We are thus able to write

$$\begin{aligned} (T_{w_\alpha} f)(g) &= \int_F f(w_\alpha^{-1} e_\alpha(x) g) dx \\ &= \int_F f(e_\alpha(1/x) x^\alpha e_{-\alpha}(-1/x) g) dx \\ &= \int_F \delta^{1/2}(x^\alpha) \pi(x^\alpha) f(e_{-\alpha}(-1/x) g) dx \end{aligned}$$

In the above, $e_\gamma(x)$ is the canonical lift from G to $\widetilde{G}$ of the one dimensional unipotent subgroup corresponding to the coroot γ, as defined in Section 1.2.

Since f is locally constant, there exists a positive number N such that for $|x| \geq N$ we have $f(e_{-\alpha}(-1/x)g) = f(g)$. Now we shall break up our integral over F into a sum of two integrals, the first over $|x| < N$ and the second over $|x| \geq N$. The first integral is an integral over a compact set so is automatically convergent. We will now study the second integral in greater detail.

Note that $\varpi^{n\alpha}$ is central in $\widetilde{T}$. We may assume without loss of generality that $f(g) \in V$ is an eigenvector of $\pi(\varpi^{n\alpha})$ with corresponding eigenvalue $(q^{-1}x_\alpha)^n$. Then our second integral becomes

$$\int_{|x|\geq N} (\delta^{1/2}\pi)(x^\alpha)f(e_{-\alpha}(-1/x)g)dx = \left(\int_{m\leq v(x)<m+n} (\delta^{1/2}\pi)(x^\alpha)f(g)\right)\left(\sum_{i=0}^{\infty} x_\alpha^{in}\right). \quad (2.3)$$

This is absolutely convergent if and only if $|x_\alpha| < 1$, proving the theorem. □

For ease of exposition, we shall now restrict ourselves to the case of intertwining operators from $I(\chi)$ to $I(\chi^w)$. Under this restriction, the complex numbers x_α are essentially well-defined, in that different choices of eigenvectors will only change them by an n-th root of unity. We may pick any such eigenvector to define the x_α, any subsequent formulae will be independent of such choices.

Define a renormalised version of the intertwining operator by

$$\widetilde{T}_w = \prod_{\substack{\alpha>0 \\ w\alpha<0}} (1 - x_\alpha^n)T_w. \quad (2.4)$$

Lemma 2.3.4. *The collection of renormalised intertwining operators $\widetilde{T}_{w_\alpha}$ satisfies the braid relations.*

Proof. Since we know that the unnormalised intertwining operators T_{w_α} satisfy the braid relations, to check this Lemma, it suffices to check that $c(w_1w_2, x) = c(w_1, w_2x)c(w_2, x)$ where $c(w, x)$ is the renormalising coefficient in (2.4). This is a triviality. □

We are now in a position to analytically continue the intertwining operators $\widetilde{T}_w$.

If λ is the eigenvalue of $\pi(\varpi^{n\alpha})$ on V, then λq^{-s} is the eigenvalue of $\pi(\varpi^{n\alpha})$ on $V \otimes \delta^s$. Then by (2.3), $(\widetilde{T}_{w_\alpha} f_s)(g)$ is a polynomial in λq^{-s}, so in particular is a holomorphic function in s. Recall that the section f_s is as defined in (2.1).

For $\Re(s)$ sufficiently large, the defining integral for $\widetilde{T}_{w_\alpha} f_s$ is absolutely convergent. Thus we can define $\widetilde{T}_{w_\alpha} f_s$ for all $s \in \mathbb{C}$ by analytic continuation. In particular, for all V, we have now defined

$$\widetilde{T}_{w_\alpha} : I(V) \longrightarrow I(V^{w_\alpha})$$

and since the maps $\widetilde{T}_{w_\alpha}$ satisfy the braid relations, we have also defined

$$\widetilde{T}_w : I(V) \longrightarrow I(V^w)$$

for all $w \in W$.

Now let us suppose that $V = i(\chi)$ is an irreducible unramified representation of $\widetilde{T}$. By Lemma 2.2.3, $I(V)$ contains a K-fixed vector. Let ϕ_K be such a vector for $I(V)$ and ϕ_K^w be such a vector for $I(V^w)$. We normalise these spherical functions such that $(\phi_K(1_{\widetilde{G}}))(1_{\widetilde{T}}) = 1$. The spherical vectors ϕ_K and ϕ_K^w are related by $\widetilde{T}_w$ in a manner given by the following theorem. The integer n_α is defined to be $\frac{n}{(n,Q(\alpha))}$ where the notation $(\cdot, \cdot)$ here is that of the greatest common divisor.

Theorem 2.3.5. *[31, Theorem 6.5]*

$$\widetilde{T}_w \phi_K = \prod_{\alpha \in \Phi_w} (1 - q^{-1} x_\alpha^{n_\alpha}) \frac{1 - x_\alpha^n}{1 - x_\alpha^{n_\alpha}} \phi_K^w.$$

Proof. The proof in [31], also covered in the following chapter, is for the case of G semisimple and simply connected, so we need to show how to reduce to this case. First we note that $T_w \phi_K$ is a priori K-fixed, so by Lemma 2.2.3, it suffices to calculate the integral

$$I_\chi = \left(\int_U \phi_K(w^{-1}u) du \right) (1_{\widetilde{T}}).$$

Consider the natural map from the corresponding simply connected semisimple group $G_{\mathrm{ss}}^{\mathrm{sc}}$ to G. We can pullback the central extension $\widetilde{G}$ of G to a central extension of $G_{\mathrm{ss}}^{\mathrm{sc}}$ and thus consider the corresponding group $H_{\mathrm{ss}}^{\mathrm{sc}}$. The character χ of H can be extended to a character χ' of $H_{\mathrm{ss}}^{\mathrm{sc}}$. In calculating I_χ, only group elements in the image of $G_{\mathrm{ss}}^{\mathrm{sc}}$ occur and we see that the calculation is the same as for the corresponding integral $I_{\chi'}$. In this way, this theorem is reduced to the semisimple, simply connected case. $\square$

Corollary 2.3.6. *For generic χ (so on a Zariski open subset of such characters), the intertwining operator $\widetilde{T}_w$ induces an isomorphism $I(\chi) \simeq I(\chi^w)$.*

Proof. The functor $\widetilde{T}_w$ restricts to a morphism from $I(\chi)^K$ to $I(\chi^w)^K$. These two spaces are one-dimensional, so we have an isomorphism as long as $\widetilde{T}_w\phi_K$ is nonzero. The Corollary now follows immediately from Theorem 2.3.5. □

At this point, we have developed the theory as far as is necessary for the purposes of the Satake isomorphism. Following the works of Casselman [14], Kazhdan-Patterson [22] and Rodier [37], one could push this line of though further to produce stronger results on the composition series of principal series representations, though we shall not do this here.

2.4 Whittaker Functions

In this section we consider (π, V) a spherical genuine admissible representation of $\widetilde{G}$. Let ψ be a character of U such that the restriction of ψ to each one dimensional subgroup U_α for α a simple coroot is non-trivial.

Let $\mathcal{W}$ denote the space of smooth functions $f : \widetilde{G} \longrightarrow \mathbb{C}$ such that $f(\zeta ng) = \zeta\psi(n)f(g)$ for $\zeta \in \mu_r$ and $n \in N$. Then a Whittaker model for (π, V) is defined to be a $\widetilde{G}$-morphism from V to $\mathcal{W}$. A Whittaker function is any non-zero spherical vector in a Whittaker model. It thus is a function $W_\chi : \widetilde{G} \longrightarrow \mathbb{C}$ satisfying

$$W_\chi(\zeta ngk) = \zeta\psi(n)W_\chi(g) \quad \text{for } \zeta \in \mu_r, n \in N, g \in G, k \in K \tag{2.1}$$

Define the twisted Jacquet functor J_ψ from $\mathrm{Rep}(\widetilde{G})$ to $\mathrm{Vect}_{\mathbb{C}}$ by $J_\psi(V) = V/V_\psi(U)$, where $V_\psi(U)$ is the subspace of V generated by the vectors $\pi(u)v - \psi(u)v$ for all $u \in U$ and $v \in V$. There is a natural bijection between $J_\psi(V)$ and the vector space of Whittaker models of V.

Theorem 5.2 of [5] can be used to compute the dimension of the space of Whittaker functions in the same manner as it was used to compute the composition series of a Jacquet module of an induced representation.

Theorem 2.4.1. *The dimension of the space of Whittaker functions for a principal series representation $I(\chi)$ is $|\widetilde{T}/H|$.*

We apply [5, Theorem 5.2] with $\mathbf{G} = \widetilde{G}$, $\mathbf{P} = \widetilde{B}$, $\mathbf{M} = \widetilde{\mathbf{T}}$, $\mathbf{U} = \mathbf{Q} = \mathbf{V} = U$, $\mathbf{N} = 1$, $\theta = 1$ and ψ non-trivial as above. Of the glued functors that appear in the composition series of $J_\psi(I(\chi))$ via [5, Theorem 5.2], only one is non-zero, and it is the forgetful functor from $\operatorname{Rep}(\widetilde{T})$ to $\mathrm{Vect}_{\mathbb{C}}$.

If f is a spherical vector in $I(\chi)$, then we can construct a Whittaker function as the integral

$$W(g) = \int_U f(w_0 ug)\psi(u)du.$$

Technically speaking, this is a $i(\chi)$-valued function, so to obtain a $\mathbb{C}$-valued Whittaker function, we should compose with a functional on $i(\chi)$. Such a choice is made in Chapter 3 where a complex-valued Whittaker function is evaluated. In fact, in Chapter 3, a basis for the space of Whittaker functions is computed together with the production of an explicit formula for $W(t)$ with $t \in \widetilde{T}$ in the case where $G = SL_n$. Note that by (2.1) and the Iwasawa decomposition, W is completely determined by its restriction to $\widetilde{T}$. There is an alternative method of Chinta and Offen [17] for calculating these metaplectic Whittaker functions. Their method more closely follows the lines of the original work of Casselman and Shalika [15], again working in type A.

Chapter 3

The Whittaker Function

3.1 An Explicit Iwasawa Decomposition

At this point in time, we take a break from the development of the representation theory and build towards developing formulae for Whittaker functions and a Gindikin-Karpelevich formula for the metaplectic group. We work with a simple simply connected group $\mathbf{G}$ and use the notation developed in Section 1.3.

In this section, we shall give an algorithm that explicitly calculates the Iwasawa decomposition in $\widetilde{G}$. This algorithm will form the cornerstone for the rest of this paper.

Let $K = G(O_F)$, a maximal compact subgroup of G. Since we have assumed that n and q are coprime, the central extension $\widetilde{G}$ of G splits over K [35, Lemma 11.3]. Thus there is a section $s: K \longrightarrow \widetilde{G}$ that is a homomorphism such that $\widetilde{K}$ is the direct product of its subgroups $s(K)$ and μ_n. We shall choose such a splitting s and identify K with its image in $\widetilde{G}$ under s when appropriate.

Let T be the subgroup of G generated by (the images of) all elements of the form $h_\alpha(x)$, and let B be the subgroup of G generated by T and the images of $e_\alpha(x)$ for all $\alpha > 0$ and $x \in F$. Then B is a Borel subgroup of G with maximal torus T. The Iwasawa decomposition [11] states that $G = BK$. This clearly lifts to $\widetilde{G}$ and so we may write $\widetilde{G} = \widetilde{B}K$.

The unipotent subgroup opposite to B in G is denoted U^-. Steinberg [42] shows that the universal central extension E splits over U^-, so our central extension $\widetilde{G}$ certainly also does. Thus we may identify U^- with a corresponding subgroup of $\widetilde{G}$. Explicitly, we have that U^- is the subgroup of $\widetilde{G}$ generated by all $e_\alpha(x)$ where $\alpha \in \Phi^-$ and $x \in F$. We shall give an algorithm that explicitly calculates the Iwasawa decomposition of any $u \in U^-$.

Lemma 3.1.1. *If $\alpha \in \Phi$ and $x \in O_F$, then $e_\alpha(x) \in s(K)$.*

Proof. Let us fix one $\alpha \in \Phi$. The group $\{e_\alpha(t) | t \in O_F\}$ is a splitting of the corresponding subgroup of K, a subgroup that is isomorphic to the additive group O_F. Any two splittings of this subgroup must differ by an element of $\mathrm{Hom}\,(O_F, \mu_n)$, which is trivial, so the splitting is unique, and thus must agree with the restriction of s, proving that $e_\alpha(t) \in s(K)$. □

We shall now recall a result regarding the action of the Weyl group on a root system.

Proposition 3.1.2. *Let $w = s_{i_1} \dots s_{i_k}$ be a reduced decomposition of $w \in W$ into simple reflections s_{i_j}. Then*

$$\{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\} = \{\alpha_{i_k}, s_{i_k}\alpha_{i_{k-1}}, s_{i_k}s_{i_{k-1}}\alpha_{i_{k-2}}, \dots, s_{i_k}s_{i_{k-1}} \dots s_{i_2}\alpha_{i_1}\}.$$

The proof may be found in any standard text on Lie theory, for example [6, Ch VI, §6, Corollaire 2].

Let $\mathbf{i} = (i_1, \dots, i_N)$ be an N-tuple of elements of I such that $w_0 = s_{i_1}s_{i_2} \dots s_{i_N}$ is a reduced decomposition of the long word w_0. Let $\mathcal{I}$ denote the set of all such tuples $\mathbf{i}$.

For any $\mathbf{i} \in \mathcal{I}$, the above theorem induces a total ordering $<_\mathbf{i}$ on the set of positive roots, given by

$$\Phi^+ = \{\alpha_1 <_\mathbf{i} \dots <_\mathbf{i} \alpha_N\} \quad \text{where} \quad \alpha_j = s_{i_N} \dots s_{i_{j+1}}\alpha_{i_j}.$$

With this notation, we have now defined α_j for $j \in I$ as well as for $j \in [N]$. We hope that this does not cause any confusion for the reader.

For each $k \in [N]$, let G_k denote the set of elements $g \in \widetilde{G}$ which can be expressed in the form

$$g = \left(\prod_{j=1}^{k-1} e_{-\alpha_j}(x_j)\right) h \left(\prod_{j=k+1}^{N} e_{\alpha_j}(x_j)\right) \tag{3.1}$$

where all $x_j \in F$ and $h \in \widetilde{T}$.

Lemma 3.1.3. *For all $z \in F$ and $g \in G_k$, there exist unique $z' \in F$ and $g' \in G_k$ such that*

$$e_{-\alpha_k}(z)g = g'e_{-\alpha_k}(z').$$

Proof. This involves repeated applications of the commutation relations presented in Section 1.3 to push the $e_{-\alpha_k}(z)$ term in $e_{-\alpha_k}(z)g$ from the left to the right hand side of the product.

It is straightforward to push a term of the form $e_{-\alpha_k}(\cdot)$ past another term of the form $e_{-\alpha_{k'}}(\cdot)$ or past a term of the form $h_\beta(\cdot)$. The key to ensuring that the manipulation claimed by the lemma is possible is thus understanding what terms appear in the commutator formula (1.1) when a term of the form $e_{-\alpha}(x)$ is pushed past a term of the form $e_\beta(y)$ for some $\alpha, \beta \in \Phi^+$, necessarily with $\alpha <_{\mathbf{i}} \beta$.

Suppose that a term of the form $e_\gamma(w)$ appears in computing such a commutator. We will show that if $\gamma \in \Phi^+$, then $\gamma >_{\mathbf{i}} \beta$ whereas if $\gamma \in \Phi^-$, then $-\gamma <_{\mathbf{i}} \alpha$. This is enough to see that after applying the commutation relations a finite number of times, we will have written $e_{-\alpha_k}(z)g$ in the desired form $g'e_{-\alpha_k}(z')$. Briefly, the reason that this suffices is the following. First, we push the $e_{-\alpha_k}(\cdot)$ term to the rightmost side of the product. Then consider the multiset consisting of all integers $i-j$ for which there is a $e_{-\alpha_i}(\cdot)$ term appearing to the left of a $e_{\alpha_j}(\cdot)$ in our product. Such occurrences are undesirable. With each application of the commutation relations, an element $j-i$ in this multiset is replaced by a finite set of integers strictly larger than $j-i$. Since the elements of this multiset are integers bounded above by N, eventually our multiset becomes empty, as desired.

So now let us suppose that γ is such that a term $e_\gamma(w)$ appears in pushing a term of the form $e_{-\alpha}(x)$ is past a term of the form $e_\beta(y)$ where $\alpha, \beta \in \Phi^+$ with $\alpha <_{\mathbf{i}} \beta$.. Then we have $\gamma = i\beta - j\alpha$ for some positive integers i and j. Write $\alpha = s_{i_N} \dots s_{i_{r+1}} \alpha_{i_r}$ and $\beta = s_{i_N} \dots s_{i_{s+1}} \alpha_{i_s}$. Note that by our assumption $\alpha <_{\mathbf{i}} \beta$ we have $r < s$.

Let $w_1 = s_{i_{r+1}} \dots s_{i_N}$. Then using Proposition 3.1.2 we obtain that $w_1(\beta) \in \Phi^-$ and $w_1(\alpha) \in \Phi^+$. Thus if $\gamma \in \Phi^-$, we have $-\gamma \in \Phi^+$ and $w_1(-\gamma) \in \Phi^+$ so we can conclude, again using Proposition 3.1.2 that $-\gamma <_{\mathbf{i}} \alpha$ as desired.

Now let $w_2 = s_{i_{s-1}} \dots s_{i-1}$. Note that $-w_0\alpha = s_{i_1} \dots s_{i_{r-1}} \alpha_r$ and similarly for β. Thus we use Proposition 3.1.2 for w_2 to obtain that $w_2(-w_0\alpha) \in \Phi^-$ and $w_2(-w_0\beta) \in \Phi^+$ so $w_2(-w_0\gamma) \in \Phi^+$. Since $\gamma \in \Phi^+$ if and only if $-w_0\gamma \in \Phi^+$ this last result implies that we can write $-w_0\gamma = s_{i_1} \dots s_{i_{t-1}} \alpha_t$ for some $t > s$, implying $\gamma = s_{i_N} \dots s_{i_{t+1}} \alpha_t$ and thus $\gamma >_{\mathbf{i}} \beta$, as required.

□

Now we are in a position to describe our algorithm for an explicit Iwasawa decomposition of $u \in U^-$. A sample computation in the case of SL_3 will be presented at the conclusion of this section. This algorithm runs as follows:

Algorithm 3.1.4. *Given $u \in U^-$, we may write u in the form*

$$u = \prod_{j=1}^{N} e_{-\alpha_j}(x_j).$$

for unique $x_1, \ldots, x_N \in F$.

For $k \in [N]$, we inductively define elements $p_k, p'_k \in \widetilde{G}$ and $y_k \in F$ as follows:

Initialise $p_{N+1} = 1_{\widetilde{G}}$. By decreasing induction on k, we use Lemma 3.1.3 to define $p'_k \in G_k$ and $y_k \in F$ such that $e_{-\alpha_k}(x_k)p_{k+1} = p'_k e_{-\alpha_k}(y_k)$.

Let

$$p_k = \begin{cases} p'_k & \text{if } |y_k| \leq 1 \\ p'_k h_{\alpha_k}(y_k^{-1}) e_\alpha(y_k) & \text{if } |y_k| > 1. \end{cases}$$

Once this computation has been completed for $k = 1$, the algorithm halts, with its primary output as the group element p_1.

In the above algorithm, variables x_i and y_i are introduced which we shall make frequent use of throughout the sequel. Both of these sets of variables are implicit functions of u and $\mathbf{i}$, however for ease of exposition, this dependence will be usually suppressed from the notation.

Theorem 3.1.5. *The above algorithm produces a well defined output that computes the Iwasawa decomposition of $u \in U^-$, in the sense that writing $u = p_1 k$, we have $p_1 \in \widetilde{B}$ and $k \in K$.*

Proof. It is a consequence of Lemma 3.1.3 that at all steps in the inductive procedure, $p_k \in G_k$ and $y_k \in F$ are uniquely determined. This is mostly straightforward, the only potential sticking point is that a priori we do not have $p_{k+1} \in G_k$. Instead we have $p_{k+1} = e_{-\alpha_k}(z_k)p''$ for some $z_k \in F$ and $p'' \in G_k$ and thus writing $e_{-\alpha_k}(x_k)p_{k+1} = e_{-\alpha_k}(x_k + z_k)p''_k$ we see that the algorithm may continue unhindered. Thus the output of this algorithm is well defined.

Note that $p_1 \in \widetilde{B}$ since $p'_1 \in G_1$. To show that $k \in K$ we need to show that if $|y_k| < 1$ then $e_{-\alpha_k}(y_k) \in K$, and secondly if $|y_k| \geq 1$ then $e_{-\alpha_k}(y_k)^{-1} h_{\alpha_k}(y_k^{-1}) e_\alpha(y_k) \in K$, as k is a product of such terms. Since we have an identity

$$e_{-\alpha}(y)^{-1} h_\alpha(y^{-1}) e_\alpha(y) = e_\alpha(y^{-1}) w_{-\alpha}(1),$$

this desired result follows from Lemma 3.1.1. $\square$

Let us define variables w_k for $k \in [N]$ as follows

$$w_k = \begin{cases} y_k & \text{if } |y_k| > 1 \\ 1 & \text{if } |y_k| \leq 1. \end{cases}$$

Then the following description of the diagonal part of p_k is immediate.

Proposition 3.1.6. *When writing p_k in the form*

$$p_k = \left(\prod_{j=1}^{k-1} e_{-\alpha_j}(t_j) \right) h \left(\prod_{j=k+1}^{N} e_{\alpha_j}(t_j) \right),$$

with $h \in \widetilde{T}$, we have

$$h = \prod_{i=N}^{k} h_{-\alpha_i}(w_i).$$

Here we are freely using the fact that $h_{-\alpha}(w) = h_\alpha(w^{-1})$.

Define integers m_k for $k \in [N]$ according to the valuation of the variables w_k by $m_k = -v(w_k)$. (The m variables, like x, y and w are functions of u and $\mathbf{i}$, again this is usually suppressed from the notation). We are now in a position to define the subsets of U^- that will be of primary concern to us.

Given any $\mathbf{i} \in \mathcal{I}$ and $\mathbf{m} = (m_1, \ldots, m_N) \in \mathbb{N}^N$, define

$$C_{\mathbf{m}}^{\mathbf{i}} = \{u \in U^- \mid m_k(u, \mathbf{i}) = m_k \text{ for all } k \in [N]\}.$$

For linguistic convenience, we shall at times refer to these sets as 'cells.' For a fixed choice of $\mathbf{i}$, the set of above cells clearly decompose U^- as a disjoint union.

We now give an example of these cells in the case where $G = SL_3$. Factorise $u \in U^-$ as

$$u = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & z & 1 \end{pmatrix}$$

where $x, y, z \in F$ and introduce the auxiliary variable $w = xz - y$. Then for a triple of non-negative integers $\mathbf{m} = (m_1, m_2, m_3)$, the cell $C_{\mathbf{m}}^{212}$ is given by the set of u as above subject to the following conditions:

$$\begin{array}{llll}
\text{If } m_1, m_2, m_3 > 0: & v(z) = -m_3 & v(\frac{y}{z}) = -m_2, & v(\frac{wz}{y}) = -m_1. \\
\text{If } m_1 = 0; m_2, m_3 > 0: & v(z) = -m_3, & v(\frac{y}{z}) = -m_2, & |\frac{wz}{y}| \leq 1. \\
\text{If } m_2 = 0; m_1, m_3 > 0: & v(z) = -m_3, & |\frac{y}{z}| \leq 1, & v(w) = -m_1. \\
\text{If } m_2 = m_1 = 0; m_3 > 0: & v(z) = -m_3, & |\frac{y}{z}| \leq 1, & |w| \leq 1. \\
\text{If } m_3 = 0; m_1, m_2 > 0: & |z| \leq 1, & v(y) = -m_2, & v(\frac{x}{y}) = -m_1. \\
\text{If } m_3 = m_1 = 0; m_2 > 0: & |z| \leq 1, & v(y) = -m_2, & |\frac{x}{y}| \leq 1. \\
\text{If } m_3 = m_2 = 0; m_1 > 0: & |z| \leq 1, & |y| \leq 1, & v(x) = -m_1. \\
\text{If } m_3 = m_2 = m_1 = 0: & |z| \leq 1, & |y| \leq 1, & |x| \leq 1.
\end{array}$$

For example, in the third case considered above, this follows from the following calculations, where we repeatedly use the rank one identity $\left(\begin{smallmatrix}1&0\\t&1\end{smallmatrix}\right) = \left(\begin{smallmatrix}t^{-1}&1\\0&t\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-1\\1&t^{-1}\end{smallmatrix}\right)$. In the subsequent list of equations, the letter k refers to an element of K but may represent different elements of K on different lines. We see that $y_3 = z$, $y_2 = y/z$ and $y_1 = xz - y$.

$$\begin{aligned}
u &= \begin{pmatrix}1&0&0\\x&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\y&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&z&1\end{pmatrix} \\
&= \begin{pmatrix}1&0&0\\x&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-y/z&1/z&1\\0&0&z\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\y/z&0&1\end{pmatrix}k \\
&= \begin{pmatrix}1&0&0\\0&1/z&1\\0&0&z\end{pmatrix}\begin{pmatrix}1&0&0\\xz-y&1&0\\0&0&1\end{pmatrix}k \\
&= \begin{pmatrix}1/(xz-y)&1&0\\0&(xz-y)/z&1\\0&0&z\end{pmatrix}k.
\end{aligned}$$

The manipulations are deceptively simple in this example, but we wished to present an example with one of the m_i variables equal to zero, as otherwise we would be repeating calculations that have been performed previously in [4].

3.2 Representation Theory and Metaplectic Whittaker Functions

We require some knowledge of unramified genuine principal series representations of $\widetilde{G}$. This is treated in [39] in the simply laced case, and in [32] in greater generality that encompasses all our needs. One can also consult [22] for a detailed study in the case $G = GL_n$. The meaning of the adjective 'genuine' is that the central μ_n must act by a fixed faithful character. We fix an inclusion $\mu_n \subset \mathbb{C}^\times$ and thus assume that for any genuine representation (π, V) of $\widetilde{G}$, we have that $\pi(\zeta g) = \zeta\pi(g)$ for all $\zeta \in \mu_n$ and $g \in \widetilde{G}$.

The group $\widetilde{T}$ is a Heisenberg group (a two-step nilpotent group). Since the commutator subgroup $[\widetilde{T}, \widetilde{T}]$ is contained in the central μ_n, by Corollary 2.1.2, any irreducible genuine representation of $\widetilde{T}$ is induced from a character of a maximal abelian subgroup of $\widetilde{T}$. Following Lemma 2.1.3, let $H = Z_{\widetilde{T}}(\widetilde{T} \cap K)$ be a maximal abelian subgroup of $\widetilde{T}$ containing $\widetilde{T} \cap K$.

Let χ be a genuine complex character of the group $H/(\widetilde{T} \cap K)$. We choose complex numbers x_i for $i \in I$ with the property that

$$\chi\left(\prod_{i\in I} h_{\alpha_i}(\varpi^{m_i})\right) = \prod_{i\in I} x_i^{m_i}$$

whenever the argument of χ above lies in H. This is always possible, the condition on $\{m_i\}_{i\in I}$ for the above product to lie in H is that m_i lie in a sublattice of $n\mathbb{Z}^I$. An explicit characterisation of this lattice is obtained in the proof of Lemma 2.1.3, though we shall not require such a description here.

Recall the definition of the principal series representations $I(\chi)$ from Section 2.2, and Lemma 2.2.3, which says that an unramified genuine principal series representation $(\pi_\chi, I(\chi))$ as above has a one-dimensional space of K-fixed vectors.

Let $\phi_K : \widetilde{G} \longrightarrow W_\chi$ be a non-zero element of $I(\chi)^K$. We define a function $f : \widetilde{G} \longrightarrow \mathbb{C}$ related to ϕ_K by the following.

$$f\left(\zeta u \prod_{i\in I} h_{\alpha_i}(\varpi^{m_i}) k\right) = \zeta \prod_{i\in I} (q^{-1} x_{\alpha_i})^{m_i}. \tag{3.1}$$

where $\zeta \in \mu_n$, $u \in U$, $m_i \in \mathbb{Z}$ and $k \in K$, using the Iwasawa decomposition to express an arbitrary $g \in \widetilde{G}$ as a product of such terms.

It is not immediately obvious that this is a well-defined function; we shall prove this as part of the following Propostion.

Proposition 3.2.1. *There exists a linear functional λ on W_χ such that $\lambda(\phi_K(g)) = f(g)$ for all $g \in \widetilde{G}$.*

Proof. Let A be the group of all elements of the form $\prod_{i \in I} h_{\alpha_i}(\varpi^{m_i})$ where $m_i \in \mathbb{Z}$. This is abelian because our assumption that $\mu_{2n} \subset F$ forces all commutators calculated from (1.4) to vanish.

Choose a set of coset representatives $a_j \in A$ for $\widetilde{T}/H$. Then the set of elements $\{\pi_\chi(a_j)\phi_K(1)\}$ forms a basis of W_χ. Thus there exists a linear functional $\lambda\colon W_\chi \longrightarrow \mathbb{C}$ such that

$$\lambda(\pi_\chi(a_j)\phi_K(1)) = \chi(a_j)$$

where $\chi\colon A \longrightarrow \mathbb{C}$ is the extension of χ on $A \cap H$ given by $\chi(\prod_{i \in I} h_{\alpha_i}(\varpi^{m_i})) = \prod_{i \in I} x_i^{m_i}$.

Now suppose we have an arbitrary $a \in A$. Then we can write $a = a_j h$ for some j and some $h \in H$ and hence

$$\begin{aligned} \lambda(\phi_K)(a) &= \lambda(\pi_\chi(a_j)\phi_K(h)) \\ &= (\delta^{1/2}\chi)(h)\lambda(\pi_\chi(a_j)\phi_K(1)) \\ &= (\delta^{1/2}\chi)(h)(\delta^{1/2}\chi)(a_i) \\ &= (\delta^{1/2}\chi)(a). \end{aligned}$$

Thus $\lambda \circ \phi_K$ and f agree on A. Since ϕ_K is a genuine function, left U-invariant and right K-invariant, this is enough to show that $\lambda \circ \phi_K = f$, thus proving the proposition, and as a consequence ensuring that the function f is well-defined. $\square$

Our main thrust of this paper involves computing the integral of f over U^-, as well as computing the integral of f against a character of U^-. We shall now explain the representation theoretic meanings behind these integrals.

The Weyl group W acts on principal series representations. As in the reductive case, we have from Theorem 2.3.1

$$\dim(\operatorname{Hom}(I(\chi), I(\chi)^{w_0})) = 1.$$

Letting T denote a non-zero element of the above space of intertwining operators, and ϕ_K and ϕ'_K be spherical vectors for $I(\chi)$ and $I(\chi)^{w_0}$ respectively, we have $T\phi_K = g(\chi)\phi'_K$ for some meromorphic function g. The representation $I(\chi)$ will be irreducible whenever g does not have a zero or a pole, and we can write g as the integral

$$\int_{U^-} f(u)du.$$

with f as in (3.1) above. It will transpire that this integral is only dependent on χ, not on the choices of x_i that were made in defining f.

Now let ψ denote a character of U^- such that the restriction to the subgroup $U_{-\alpha} = \{e_{-\alpha}(x) \mid x \in F\} \cong F$ for each simple α has conductor O_F. As in the reductive case, the integral

$$W(g) = \int_{U^-} \phi_K(ug)\psi(u)du$$

defines a Whittaker function; it is a spherical vector in a Whittaker model for V. The above integral actually defines a W_χ-valued Whittaker function, so to obtain a complex valued Whittaker function, we merely need to compose with a linear functional λ, and thus we are lead to study the integral

$$\int_{U^-} f(ug)\psi(u)du.$$

Since a Whittaker function transforms on the left under U by ψ, on the right under K trivially, and is genuine, a Whittaker function is determined by the values it takes on the maximal torus T, by the Iwasawa decomposition. If $g \in T$, making a change of coordinate in the above integral, replacing u by gug^{-1}, means that one is instead lead to evaluating the integral

$$\int_{U^-} f(u)\psi(gug^{-1})du$$

where a relatively trivial constant factor has been discarded.

The lattice Λ is the coroot lattice of the corresponding adjoint group G^{ad}, and thus acts by conjugation on U^- and hence on the set of characters of U^-. Explicitly, the action of $\lambda \in \Lambda$ on our fixed character ψ creates another character ψ_λ defined by

$$\psi_\lambda(u) = \psi(\varpi^\lambda u \varpi^{-\lambda}).$$

Here, although ϖ^λ is not in G, it is in G^{ad} which canonically shares the same unipotent subgroups as G, so acts on U^- by conjugation.

In this language, explicitly evaluating a Whittaker function becomes equivalent to the problem of evaluating the integral

$$I_\lambda = \int_{U^-} f(u)\psi_\lambda(u)du. \tag{3.2}$$

For the remainder of this paper, our major goal is to obtain a method for evaluating the above integral.

There is a simple case where we know that this integral vanishes. This vanishing property, together with its proof, is exactly the same as in the reductive case, as shown in [15, Lemma 5.1].

Proposition 3.2.2. *Unless λ is dominant, we have $I_\lambda = 0$.*

3.3 Initial Calculations with the Cell Decomposition

In this section we shall present some initial calculations involving the decomposition of U^- into the disjoint union

$$U^- = \bigsqcup_{\mathbf{m}\in\mathbb{N}^N} C^{\mathbf{i}}_{\mathbf{m}}.$$

This will immediately result in a metaplectic Gindikin-Karpelevic formula, generalising the type A results of Bump and Nakasuji [13]. The intermediate results we obtain will also prove to be useful in the sequel when we are evaluating the Whittaker function.

We choose a Haar measure on F such that O_F has volume 1, and denote it by dx. Now choose a normalisation of Haar measure on U^- such that $du = \prod_{i=1}^N dx_i$. The variables x_i here that we use are those introduced in Algorithm 1, we also freely use the y_i and w_i variables defined in the main algorithm (which all depend on a choice of $\mathbf{i}$). Under this normalisation, we can now compute the volume of the sets $C^{\mathbf{i}}_{\mathbf{m}}$.

Proposition 3.3.1. *The volume of $C^{\mathbf{i}}_{\mathbf{m}}$ is*

$$\prod_{i=1}^N q^{\langle\rho,\alpha_i^\vee\rangle m_i}\left(1-\frac{1-\delta_{m_i,0}}{q}\right)$$

(δ here represents the Kronecker delta function).

Proof. We wish to calculate $\int_{C^{\mathbf{i}}_{\mathbf{m}}} 1\ dx_1 \dots dx_N$ and it is easy to see that

$$\int_{C^{\mathbf{i}}_{\mathbf{m}}} 1\ dy_1 \dots dy_N = \prod_{i=1}^{N} q^{m_i}(1 - \frac{1-\delta_{m_i,0}}{q}).$$

So we need to calculate the Jacobian for the change of coordinates from $x_1, \dots, x_N$ to $y_1, \dots, y_N$, the formulae for which are obtained as a byproduct of Algorithm 1.

In our algorithm, consider what happens to the argument of the term $e_{-\alpha_k}(\cdot)$ as $e_{-\alpha_k}(x_k)$ is pushed passed all other terms in p_{k+1} to the right hand side of the product, where it emerges as $e_{-\alpha_k}(y_k)$. Apart from the initial step where a constant (that is independent of $x_k, \dots, x_N$) is added to the argument, the only time when the argument of $e_{-\alpha_k}(\cdot)$ changes is when it moves past a term of the form $h_{-\beta}(w_\beta)$, which has the effect of multiplying the argument by $w_k^{-\langle \alpha_k, \beta^\vee \rangle}$. Here we are using (1.2) as well as Proposition 3.1.6. Necessarily, we must have that $\beta >_{\mathbf{i}} \alpha$ for this to occur.

Thus we have

$$dy_k = \left(\prod_{i=k+1}^{N} |w_{\alpha_i}|^{-\langle \alpha_k, \alpha_i^\vee \rangle} \right) dx_k.$$

Multiplying over all k yields

$$dy_1 \dots dy_N = \left(\prod_{i=1}^{N} q^{-m_i \sum_{j=1}^{i-1} \langle \alpha_j, \alpha_i^\vee \rangle} \right) dx_1 \dots dx_N.$$

Thus the proposition follows from the following lemma. □

Lemma 3.3.2. *For all $\beta \in \Phi^+$, we have*

$$\langle \rho, \beta^\vee \rangle = 1 + \sum_{\alpha <_{\mathbf{i}} \beta} \langle \alpha, \beta^\vee \rangle.$$

Proof. We first show that the right hand side of the above equation is independent of the choice of $\mathbf{i} \in \mathcal{I}$. To achieve this, it suffices to show invariance under changing $\mathbf{i}$ to $\mathbf{i}'$ by the application of a single braid relation. For such $\mathbf{i}$ and $\mathbf{i}'$, we have that $\alpha_j(\mathbf{i}) = \alpha_j(\mathbf{i}')$ (here we are merely expressing the dependence of α_j on the choice of $\mathbf{i}$) unless $\mathbf{i}_j \neq \mathbf{i}'_j$ so we have quickly reduced to showing invariance of long word decomposition in the rank 2 case. This is achieved by a simple case by case analysis, which we shall omit here.

Now consider two long word decompositions $\mathbf{i} = (i_1, \dots, i_N)$ and $\mathbf{i}' = (\tau(i_N), \dots, \tau(i_1))$

where $\tau : I \longrightarrow I$ is the automorphism defined by $w_0\alpha_i = -\alpha_{\tau(i)}$.

This pair of decompositions has the property that for any two $\alpha, \beta \in \Phi^+$, $\alpha <_{\mathbf{i}} \beta$ if and only if $\beta <_{\mathbf{i}'} \alpha$. Thus

$$\begin{aligned} 1 + \sum_{\alpha <_{\mathbf{i}} \beta} \langle \alpha, \beta^\vee \rangle &= \frac{1}{2}\left(1 + \sum_{\alpha <_{\mathbf{i}} \beta} \langle \alpha, \beta^\vee \rangle + 1 + \sum_{\alpha <_{\mathbf{i}'} \beta} \langle \alpha, \beta^\vee \rangle\right) \\ &= \frac{1}{2} \sum_{\alpha \in \Phi^+} \langle \alpha, \beta^\vee \rangle \\ &= \langle \rho, \beta^\vee \rangle \end{aligned}$$

as required. $\square$

Recall that n denotes the degree of the cover $\widetilde{G} \to G$. For each root α, define the integer $n_\alpha = \frac{n}{(n, l_\alpha)}$, where l_α is, as before given by $||\alpha^\vee||^2$, where the norm on the coroot lattice has been chosen such that the short coroots have length 1. We also denote $n_k = n_{\alpha_k}$. The following lemma will allow us to integrate the function f from (3.1) over any set $C^{\mathbf{i}}_{\mathbf{m}}$.

Lemma 3.3.3. *1. For the function f defined in (3.1),*

$$\int_{C^{\mathbf{i}}_{\mathbf{m}}} f(u)du = 0$$

unless $n_k | m_k$ for all $k \in [N]$.

2. If $n_k | m_k$ for all $k \in [N]$ then f is constant on $C^{\mathbf{i}}_{\mathbf{m}}$ and takes the value

$$\prod_{\alpha \in \Phi^+} (q^{\langle \rho, \alpha^\vee \rangle} x_\alpha)^{m_\alpha}$$

where the variables x_α are defined by $x_\alpha = \prod_{i \in I} x_i^{h_i}$ with $\alpha^\vee = \sum_{i \in I} h_i \alpha_i^\vee$.

Proof. By induction on k, we shall prove that

$$\int_{C^{\mathbf{i}}_{\mathbf{m}}} f(u)du \neq 0 \implies n_i | m_i \text{ for all } i \leq k$$

and in this case

$$f(\prod_{i=N}^{1} h_{-\alpha_i}(w_i)) = f\left(\prod_{i=N}^{k+1} h_{-\alpha_i}(w_i) \prod_{i=k}^{1} h_{\alpha_i}(\varpi^{m_i})\right).$$

Note that the lemma follows from the $k = N$ case of the inductive statement.

Let $w_i = \varpi^{-m_i} u_i$. Then $u_i \in O_F$. We may write, assuming our inductive hypothesis for k and using (1.4),

$$h_{-\alpha_{k+1}}(w_{k+1}) \prod_{i=k}^{1} h_{\alpha_i}(\varpi^{m_i}) = (\varpi, u_{k+1})^e \left(\prod_{i=k+1}^{1} h_{\alpha_i}(\varpi^{m_i}) \right) h_{-\alpha_{k+1}}(u_{k+1}),$$

where the exponent e is given by

$$e = m_{k+1} l_{k+1} + \sum_{i \leq k} \langle \alpha_i, \alpha_{k+1}^\vee \rangle m_i l_{\alpha_i}. \tag{3.1}$$

Consider the evaluation of the integral of f over $C_{\mathbf{m}}^{\mathbf{i}}$, by integrating over variables $y_1, \ldots, y_n$ in that order. Suppose also that $m_{k+1} > 0$. Then the above calculation shows that we get a factor of

$$\int_{O_F^\times} (\varpi, u)^e du$$

appearing in our integral, from the integration over y_{k+1}. For this integral to be non-zero, we must thus have that $n|e$. By our inductive hypothesis we have that $e \equiv m_{k+1} l_{k+1} \pmod{n}$, so we obtain the first part of the inductive hypothesis, namely that n_{k+1} divides m_{k+1}. The second part of the inductive hypothesis now follows from the form taken by f in (3.1), noting that $e \equiv 0 \pmod{n}$ and $h_{-\alpha_{k+1}}(u_{k+1}) \in K$, thus proving our lemma. $\square$

At this point it is straightforward to complete the evaluation of the integral of f over U^-. In order for this integral to be convergent, we must assume that $|x_i| < 1$ for all $i \in I$. However some sense can be made of this integral for an arbitrary collection of complex numbers x_i by considering the integral as the analytic continuation of an intertwining operator, as in Section 2.3. We obtain the following result, where we assume $|x_\alpha| < 1$ for all α in order to ensure convergence.

Theorem 3.3.4. *[Gindikin-Karpelevich Formula] For the function f defined in (3.1), and the variables x_α as defined in Lemma 3.3.3, we have*

$$\int_{U^-} f(u) du = \prod_{\alpha \in \Phi^+} \left(\frac{1 - q^{-1} x_\alpha^{n_\alpha}}{1 - x_\alpha^{n_\alpha}} \right)$$

Proof. The idea is to simply write the integral as a sum over each $C_{\mathbf{m}}^{\mathbf{i}}$, and use what we have already computed.

$$\begin{aligned}
\int_{U^-} f(u)du &= \sum_{\mathbf{m}\in\mathbb{N}^N} \int_{C^{\mathbf{i}}_{\mathbf{m}}} f(u)du = \sum_{\substack{\mathbf{m}\in\mathbb{N}^N \\ n_j|m_j\forall j}} \int_{C^{\mathbf{i}}_{\mathbf{m}}} f(u)du \\
&= \sum_{\substack{\mathbf{m}\in\mathbb{N}^N \\ n_j|m_j\forall j}} \prod_{\alpha\in\Phi^+} (q^{-\langle\rho,\alpha^\vee\rangle} x_\alpha)^{m_\alpha} \prod_{i=1}^{N} q^{\langle\rho,\alpha_i^\vee\rangle m_i}(1 - \frac{1-\delta_{m_i,0}}{q}) \\
&= \prod_{\alpha\in\Phi^+} \sum_{k=1}^{\infty} x_\alpha^{kn_\alpha}(1 - \frac{1-\delta_{k,0}}{q}) \\
&= \prod_{\alpha\in\Phi^+} \left(\frac{1-q^{-1}x_\alpha^{n_\alpha}}{1-x_\alpha^{n_\alpha}}\right)
\end{aligned}$$

as required. □

This same method can be used to calculate the integral of f over other unipotent subgroups of U^-. For each $w \in W$ let $\Phi_w = \{\alpha \in \Phi^+ \mid w\alpha \in \Phi^-\}$ and U_w^- be the corresponding unipotent subgroup. Then with exactly the same proof, we have

Theorem 3.3.5. *For the function f defined in (3.1), and the variables x_α as defined in Lemma 3.3.3, we have*

$$\int_{U_w^-} f(u)du = \prod_{\alpha\in\Phi_w} \left(\frac{1-q^{-1}x_\alpha^{n_\alpha}}{1-x_\alpha^{n_\alpha}}\right)$$

From a representation-theoretic point of view, this integral corresponds to the evaluation of the intertwining operator $T_w \in \mathrm{Hom}\,(I(\chi), I(\chi)^w)$ at a spherical vector.

3.4 Connection to the combinatorics of crystals

In this section we shall give a bijection between the sets $C^{\mathbf{i}}_{\mathbf{m}}$ that decompose U^- and Lusztig's canonical basis. We shall be considering the case $F = k((\varpi))$ for k an algebraically closed field, since this will make our exposition easier. Under this assumption, our sets $C^{\mathbf{i}}_{\mathbf{m}}$ are naturally affine varieties over k.

Although such a choice of F is not a local field, we can obtain a direct link between the case of a positive characteristic local field, which is necessarily (non-canonically) isomorphic to $\mathbb{F}_q((\varpi))$. The sets $C^{\mathbf{i}}_{\mathbf{m}}$ for the field $\mathbb{F}_q((\varpi))$ are the $\mathbb{F}_q$ points of the corresponding varieties over $\overline{\mathbb{F}_q}$ that are considered in this section. To establish a link in the case for an arbitrary discrete valuation field F with the techniques of this section, we note that a formal analogy can be made as the combinatorics of taking valuations is independent of the field F, and is an analogue of the process of tropicalisation.

Lusztig [27] realised the canonical basis as the set of connected components of a graph structure on $\mathcal{I} \times \mathbb{N}^N$, which we shall now recall. In order for there to exist an edge between two vertices $(\mathbf{i}, \mathbf{m})$ and $(\mathbf{i}', \mathbf{m}')$ it is first necessary to have $\mathbf{i}$ and $\mathbf{i}'$ related by a single application of a braid relation, so an occurrence of $(i, j, \ldots)$ is replaced by $(j, i, \ldots)$ when moving from $\mathbf{i}$ to $\mathbf{i}'$. Furthermore, we require that $\mathbf{m}$ and $\mathbf{m}'$ are related by piecewise linear transition maps $R_{\mathbf{i}}^{\mathbf{i}'}$, which we shall now explicitly describe. These equations appear in [27, 20] except for the case of type G_2, where they are derived from Section 7 of [4].

The piecewise linear transition maps $R_{\mathbf{i}}^{\mathbf{i}'}$ only affect the coordinates which are changed in moving from $\mathbf{i}$ to $\mathbf{i}'$. The restriction to the set of such coordinates is a local transtition map $R_{ij\ldots}^{ji\ldots}$, which we shall now give. There are four cases to consider, corresponding to each of the four finite type rank 2 root systems.

1. Type $A_1 \times A_1$: If $a_{ij} = a_{ji} = 0$, then

$$R_{ij}^{ji}(a, b) = (b, a).$$

2. Type A_2: If $a_{ij} = a_{ji} = -1$, then

$$R_{iji}^{jij}(a, b, c) = (b + c - \min(a, c), \min(a, c), b + a - \min(a, c)).$$

3. Type B_2: If $a_{ij} = -2$ and $a_{ji} = -1$, then

$$R_{ijij}^{jiji}(a, b, c, d) = (b + 2c + d - q, q - p, 2p - q, a + b + c - p)$$

where $p = \min(a + b, a + d, c + d)$ and $q = \min(2a + b, 2a + d, 2c + d)$.

4. Type G_2: If $a_{ij} = -3$ and $a_{ji} = -1$ then

$$R_{ijijij}^{jijiji}(a, b, c, d, e, f) = (b + 3c + 2d + 3c + f - r, r - q, 2q - r - s, s - p - q,$$
$$3p - s, a + b + 2c + d + e - p)$$

where $p = \min(a+b+2c+d, a+b+2c+f, a+b+2e+f, a+d+2e+f, b+d+2e+f)$,

$q = \min(2a + 2b + 3c + d, 2a + 2b + 3c + f, 2a + 2b + 3e + f, 2a + 2d + 3e + f, 2c + 2d + 3e + f, a + b + d + 2e + f + \min(a + c, 2c, c + e, a + e))$,

$r = \min(3a + 2b + 3c + d, 3a + 2b + 3c + f, 3a + 2b + 3e + f, 3a + 2d + 3e + f, 3c + 2d + 3e + f, 2a + b + d + 2e + f + \min(a + c, 2c, c + e, a + e))$, and

$s = \min(2a + 2b + 2c + d + \min(a + b + 3c + d, a + b + 3c + f, a + b + 3e + f, d + 2e + f + \min(a + c, 2c, c + e, a + e)) + 2f + 3\min(a + b + 2c, a + b + 2e, a + d + 2e, c + d + 2e))$.

We shall refer to this graph as the Lusztig graph.

The set of connected components of the Lusztig graph is canonically identified with the canonical basis $B(-\infty)$ for $U_q(\mathfrak{n}^+)$, the positive part of the quantised universal enveloping algebra, as defined for example in [21]. Moreover, the following theorem from [28, Ch 42] yields a parametrisation of the canonical basis for each choice of $\mathbf{i} \in \mathcal{I}$.

Theorem 3.4.1 ([28], Ch 42). *For each $\mathbf{i} \in \mathcal{I}$, the map $\mathbb{N}^N \to \mathcal{I} \times \mathbb{N}^N$ sending $\mathbf{m}$ to $(\mathbf{i}, \mathbf{m})$ is a bijection between $\mathbb{N}^N$ and the set of connected components of the Lusztig graph.*

We consider U^- as an ind-variety over k, and thus are able to give it the Zariski topology. The cells $C^{\mathbf{i}}_{\mathbf{m}}$ now have the additional structure as subvarieties of U^- and we denote their Zariski closure in U^- by $\overline{C^{\mathbf{i}}_{\mathbf{m}}}$. We are able to relate our decomposition of U^- with the canonical basis via the following theorem.

Theorem 3.4.2 (Theorem 15, [3], Theorem 4.5, [20]). *For $(\mathbf{i}, \mathbf{m}), (\mathbf{i}', \mathbf{m}') \in \mathcal{I} \times \mathbb{N}^N$, the vertices $(\mathbf{i}, \mathbf{m})$ and $(\mathbf{i}', \mathbf{m}')$ of the Lusztig graph are connected if and only if $\overline{C^{\mathbf{i}}_{\mathbf{m}}} = \overline{C^{\mathbf{i}'}_{\mathbf{m}'}}$.*

The proof of this theorem utilises the circle of ideas presented in [4], which tells one how the coordinates on U^- change when one changes the reduced word. This reduces the problem to the rank two case where it is dealt with by an explicit computation.

Corollary 3.4.3. *Consider the collection of closed subsets of U^- consisting of the set of all $\overline{C^{\mathbf{i}}_{\mathbf{m}}}$ for a fixed $\mathbf{i}$ with $\mathbf{m}$ running over $\mathbb{N}^N$. Then this collection is independent of the choice of $\mathbf{i}$.*

To proceed, we now recall some of the geometry of the affine Grassmannian for the adjoint group G^{ad}. This affine Grassmannian shall be denoted $\mathcal{G}$, it is defined to be the fpqc quotient $G^{\mathrm{ad}}(k((\varpi)))/G^{\mathrm{ad}}(k[[\varpi]])$. However, it will suffice for our purposes to consider $\mathcal{G}$ as an ind-variety over k. Recall that Λ is the cocharacter group of the maximal torus

T^{ad} of G^{ad}. The element ϖ is a uniformiser, accordingly we use ϖ^λ for $\lambda \in \Lambda$ to denote the image of ϖ in T^{ad} under the cocharacter λ.

The cells $C^{\mathbf{i}}_{\mathbf{m}}$ that provided the decomposition of $U^-(F)$ are each (non-Noetherian) affine varieties over k. This is because each $C^{\mathbf{i}}_{\mathbf{m}}$ is isomorphic to a product of copies of $k[[\varpi]]$ and $k[[\varpi]]^\times$, each of which are irreducible varieties. In particular, we see that $C^{\mathbf{i}}_{\mathbf{m}}$ is irreducible.

We first consider the action of the group $G(O_F)$ on $\mathcal{G}$ by left multiplication. The orbits are indexed by dominant weights $\lambda \in \Lambda$. More precisely they are given by $\mathcal{G}_\lambda = G(O_F)\varpi^\lambda$ according to the Cartan decomposition. The orbit $\mathcal{G}_\lambda$ is finite dimensional and its closure $\overline{\mathcal{G}}_\lambda$ is a (generally singular) projective variety.

We next consider the action of the unipotent groups $U^\pm(F)$ on $\mathcal{G}$, again by left multiplication. The orbits here are given by the Iwasawa decomposition, indexed by elements of Λ. For $\mu \in \Lambda$, we denote by $S^\pm_\mu$ the orbit $U^\pm(F)\varpi^\mu$. These strata are locally closed, but not finite dimensional.

Fix a dominant weight λ. Then there is a set of Mirkovic-Vilonen cycles (hereafter MV cycles) $\mathcal{L}(\lambda)$, defined to be the irreducible components of $\overline{S^+_\mu \cap \mathcal{G}_\lambda}$, where μ can be any weight. The only weights for which this intersection is non-empty are those for which μ belongs to the convex hull of the set $\{w\lambda \mid w \in W\}$, and such that $\lambda - \mu$ is an integral sum of coroots. As a consequence, we deduce that $\mathcal{L}(\lambda)$ is a finite set.

There is also a notion of a stable MV cycle which we now define. Let $\mathcal{L}$ denote the set of all irreducible components of $\overline{S^+_\lambda \cap S^-_\mu}$, where λ and μ run over X with $\lambda \geq \mu$. This set $\mathcal{L}$ is equipped with an action of Λ, given by $\nu Z = t^\nu Z$. We define the set of stable MV cycles to be the quotient of $\mathcal{L}$ by this action, and denote it by $\mathcal{L}(-\infty)$. We shall frequently identify $\mathcal{L}(-\infty)$ with a particular choice of coset representative in $\mathcal{L}$, most often that which is an irreducible component of $\overline{S^+_\lambda \cap S^-_0}$.

The following proposition of Anderson relates the notion of a MV cycle to that of a stable MV cycle.

Proposition 3.4.4 (Proposition 3, [1]). *Suppose λ is antidominant and ν is arbitrary. Then the irreducible components of $\overline{S^+_\nu \cap \mathcal{G}_\lambda}$ are the same as the irreducible components of $\overline{S^+_\nu \cap S^-_\lambda}$ contained in $\overline{\mathcal{G}_\lambda}$.*

In particular we are able to identify $\mathcal{L}(\lambda)$ as a subset of $\mathcal{L}(-\infty)$.

Under the geometric Satake correspondence, the intersection cohomology complex of $\overline{\mathcal{G}}_\lambda$ is mapped to the space of the representation of $(G^{\mathrm{ad}})^\vee$ (the Langlands dual to G^{ad}) of highest weight λ. Since a basis for the intersection cohomology of $\overline{\mathcal{G}}_\lambda$ is given by the set of MV cycles [33], it is natural to ask whether we can equip this set of MV cycles with the structure of a crystal. Such a procedure is carried out by Braverman and Gaitsgory in [8] for the finite crystal $\mathcal{L}(\lambda)$ and by Braverman, Finkelberg and Gaitsgory in [7] for the infinite crystal $\mathcal{L}(-\infty)$. These two crystal structures are compatible with the inclusion $\mathcal{L}(\lambda) \hookrightarrow \mathcal{L}(-\infty)$.

The following theorem gives the relation between our cells and MV cycles.

Theorem 3.4.5. *$\overline{\phi_0(C^{\mathbf{i}}_{\mathbf{m}})}$ is a MV cycle. Furthermore this map defines a bijection between $\mathbb{N}^N$ and $\mathcal{L}(-\infty)$ for any choice of $\mathbf{i} \in \mathcal{I}$.*

Before we embark on a proof, we first pause to define some useful notation.

For $\mu \in \Lambda$, define the map

$$\phi_\mu : U^- \longrightarrow \mathcal{G}, \quad \phi_\mu(u) = \varpi^\mu u.$$

For any $\mathbf{m} \in \mathbb{N}^N$ and a fixed choice of $\mathbf{i} \in \mathcal{I}$, we define $\mathbf{m}^\vee = \sum_{j=1}^N m_j \alpha_{i_j}^\vee$.

For any $\lambda \in \mathbb{Z}\Phi^\vee$, let d_λ be the number of ways of writing $\lambda = \sum_{\alpha \in \Phi^+} m_\alpha \alpha^\vee$ for non-negative integral m_α. This is a well known function, commonly referred to as the Kostant partition function. It is equal to the dimension of the λ weight space in a Verma module, and thus we know that it is equal to the number of elements of weight λ in $\mathcal{L}(-\infty)$, or equivalently the number of irreducible components of $\overline{S^+_\lambda \cap S^-_0}$.

Now we present a proof of Theorem 3.4.5.

Proof. Fix λ, and consider only those $\mathbf{m}$ for which $\mathbf{m}^\vee = \lambda$. Note that $\phi_0(C^{\mathbf{i}}_{\mathbf{m}}) \subset S^+_\lambda \cap S^-_0$. Being irreducible, its closure must lie in a stable MV cycle. Since the number of such $\mathbf{m}$ is equal to the number of such stable MV cycles (they are both equal to d_λ), and ϕ_0 of the union of such $C^{\mathbf{i}}_{\mathbf{m}}$ surjects onto $S^+_\lambda \cap S^-_0$, the closure must equal a stable MV cycle. □

A crystal $\mathcal{B}$ comes equipped with five functions from $\mathcal{B}$ which are required to satisfy various compatibility relations that can be found in [21]. The only two of these functions we will need are the Kashiwara operator $\tilde{e}_i : B \longrightarrow B \sqcup \{0\}$ and the function $\epsilon_i : B \longrightarrow$

$\mathbb{Z}\sqcup\{-\infty\}$. The crystal $B(-\infty)$ is a lowest weight crystal, it is generated by a single element $b_0 \in B(-\infty)$ together with its successive images under the various Kashiwara operators $\tilde{e}_i$.

We shall require the following proposition from [3] which identifies $\tilde{e}_i(Z)$ and $\epsilon_i(Z)$ for a MV cycle Z.

Proposition 3.4.6 (Proposition 14, [3]). *Let Z be a MV cycle, and $i \in I$. Then for each $p \in O_F$, the action of $e_{-\alpha_i}(pt^{\epsilon_i(Z)})$ stabilises Z. The MV cycle $\tilde{e}_i(Z)$ is the closure of*

$$\{e_{-\alpha_i}(x)z \mid z \in Z \text{ and } x \in F^\times \text{ such that } v(x) = \epsilon_i(Z) + 1\}.$$

We will write $B(\lambda)$ and $B(-\infty)$ for the abstract crystals reaslised by $\mathcal{L}(\lambda)$ and $\mathcal{L}(-\infty)$ respectively. For $b \in B(-\infty)$, represented by $(\mathbf{i}, \mathbf{m})$, let $C_b = \overline{C^{\mathbf{i}}_{\mathbf{m}}}$. We now move to studying the finite crystal $B(\lambda)$, for which our starting point is the following Proposition.

Proposition 3.4.7 (Proposition 8.2, [21], §8 [26], Corollary 3.4, [4], Theorem 8.4, [20]). *The image of the natural injection $B(\lambda) \hookrightarrow B(-\infty)$ is given by*

$$B(\lambda) = \{b \in B(-\infty) \mid \epsilon_i(b) + \langle \alpha_i, \lambda \rangle \geq 0 \ \forall \ i \in I\}.$$

We use this to identify $B(\lambda)$ in terms of cells C_b. There is an alternative characterisation of the cells of the finite crystal $B(\lambda)$ which will prove to be useful in the following section. For $u \in U^-$ and $i \in I$, we define $f_i(u)$ to be element of F such that in writing $u = \prod_{\alpha\in\Phi} e_{-\alpha}(z_\alpha)$, we have $z_{\alpha_i} = f_i(u)$. This result can be considered as an analogue to Theorems 8.3 and 8.5 of [20].

Theorem 3.4.8. *Suppose that λ is dominant and write $\lambda' = w_0\lambda$. Then the following three conditions on an element b of the crystal basis are equivalent.*

1. *$b \in B(\lambda)$*

2. *$\varpi^\lambda C_b \varpi^{-\lambda} \subset K$*

3. *For all $i \in I$ and $u \in C_b$, we have $f_i(\varpi^\lambda u \varpi^{-\lambda}) \in O_F$.*

Proof. $1 \Rightarrow 2$. Proposition 3.4.4 shows that $b \in B(\lambda)$ if and only if $\phi_{\lambda'}(C_b) \subset \overline{\mathcal{G}_\lambda}$, where $\lambda' = w_0\lambda$ is the image of λ under the action of the longest word in W. So if $b \in B(\lambda)$, we have $C_b \subset \phi_{\lambda'}^{-1}(\overline{\mathcal{G}_\lambda})$. It is shown in [11, 4.4.4(ii)] that $\phi_{\lambda'}^{-1}(\overline{\mathcal{G}_\lambda}) = \varpi^{-\lambda} K \varpi^{\lambda} \cap U^-$, completing this implication

$2 \Rightarrow 3$. This is immediate.

$3 \Rightarrow 1$. Suppose for all $i \in I$ and $u \in C_b$ that $f_i(\varpi^\lambda u \varpi^{-\lambda}) \in O_F$. Then $x_i(u) \in \varpi^{-\langle \alpha_i, \lambda \rangle} O_F$. So utilising Proposition 3.4.6, we have that the action of $e_{-\alpha_i}(\varpi^{\epsilon_i(b)} O_F)$ stabilising b implies that $\epsilon_i(b) \geq -\langle \alpha_i, \lambda \rangle$ which by the characterisation of $B(\lambda)$ as a subset of $B(-\infty)$ in Proposition 3.4.7 tells us that $b \in B(\lambda)$.

□

3.5 Computation of a Whittaker Function in Type A

We conclude by presenting a sample calculation of a metaplectic Whittaker function, and show that in this case, it is equal to the p-part of a Weyl Group Multiple Dirichlet Series, as defined and studied in [10]. This identity will be more than an equality of functions, as we will show that for our method of writing a Whittaker function as a weighted sum over a crystal, we will recover exactly the same weighted sum that appears in [10]. We begin by reproducing the combinatorial description of the p-part of a type A_r multiple Dirichlet series.

A Gelfand-Tsetlin pattern is a triangular array of natural numbers

$$\mathfrak{T} = \left\{ \begin{array}{ccccccccccc} a_{0,0} & & a_{0,1} & & a_{0,2} & & \cdots & & a_{0,r-1} & & a_{0,r} \\ & a_{1,1} & & a_{1,2} & & a_{1,3} & \cdots & & & a_{1,r} & \\ & & \ddots & & & & & & \iddots & & \\ & & & & & a_{r,r} & & & & & \end{array} \right\}$$

subject to the inequalities $a_{i,j} \geq a_{i+1,j+1} \geq a_{i,j+1}$ for all i and j.

To such a pattern, one attaches integers $e_{i,j}$ for all $1 \leq i \leq j \leq r$ defined by

$$e_{i,j} = \sum_{k=j}^{r} a_{i,k} - a_{i-1,k},$$

and a weight

$$G(\mathfrak{T}) = \prod_{1 \le i \le j \le r} \gamma(a_{i,j}), \quad \gamma(a_{i,j}) = \begin{cases} q^{e_{i,j}} & \text{if } a_{i-1,j-1} > a_{i,j} = a_{i-1,j}, \\ g(e_{i,j}, 0) & \text{if } a_{i-1,j-1} > a_{i,j} > a_{i-1,j}, \\ g(e_{i,j}, -1) & \text{if } a_{i-1,j-1} = a_{i,j} > a_{i-1,j}, \\ 0 & \text{if } a_{i-1,j-1} = a_{i,j} = a_{i-1,j}, \end{cases}$$

where the Gauss sums $g(a, b)$ are as defined by (1.1) (in Section 1.2). Furthermore, define integers

$$k_i(\mathfrak{T}) = \sum_{j=1}^{r} (a_{i,j} - a_{0,j}).$$

Then the p-part of a type A multiple Dirichlet series is given by the following expression:

$$\sum_{\mathfrak{T}} G(\mathfrak{T})\, q^{-2(k_1 s_1 + \ldots + k_r s_r)}$$

where the sum is over all Gelfand-Tsetlin patterns with a fixed top row and $s_1, \ldots, s_r \in \mathbb{C}$ is a collection of complex variables.

We shall work with the simple group $G = SL_{r+1}$ with the nicest possible choice of the long word decomposition. Explicitly realise

$$\Phi^+ = \{(i, j) \in [r+1] \times [r+1] \mid i < j\}.$$

Write s_i for the simple reflection corresponding to the simple root $(i, i+1)$. Then we choose $\mathbf{i} \in \mathcal{I}$ corresponding to the long word decomposition $w_0 = s_1(s_2 s_1)(s_3 s_2 s_1) \ldots (s_r s_{r-1} \ldots s_2 s_1)$. We shall refer to this word as the Gelfand-Tsetlin word. This choice of word determines the ordering $<_{\mathbf{i}}$ which can be explicitly stated in the form $(i, j) <_{\mathbf{i}} (i', j')$ if $i < i'$ or if $i = i'$ and $j < j'$.

Note that a particular feature of the Gelfand-Tsetlin word is that in performing Algorithm 3.1.4, the Iwasawa decomposition for the lower right copy of SL_r in SL_{r+1} is performed first as a substep of performing the Iwasawa decomposition for SL_{r+1}. This enables us to perform induction on r, a feature we shall frequently exploit below.

For a more in depth study of our main algorithm, we shall explicitly realise G as a $(r+1) \times (r+1)$ matrix group in the usual way. Thus $e_{(i,j)}(x)$ has an x appearing in the entry of the i-th row and j-th column, ones along the diagonal and zeroes elsewhere.

Firstly, we shall require further information regarding the top row of the matrix p_1 that appears in our Iwasawa decomposition.

Lemma 3.5.1. *The entries in the top row of p_1 are given by*

$$(p_1)_{1j} = \begin{cases} \prod_{i=j}^{r} \frac{1}{w_{1,j+1}}, & \text{if } |y_{1,j}| > 1 \text{ or } j = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The proof uses Proposition 3.1.6, which in particular proves the lemma in the $j = 1$ case, so we may assume $j > 1$. By induction on r, we know $(p_{r+1})_{1j} = 0$. Note that the only time that the i, j-entry of the p-matrix in the algorithm is altered is when a multiplication by $h_{1,j}(y_{1,j}^{-1})e_{1,j}(y_{1,j})$ occurs. Hence, $(p_1)_{ij} = (p_j)_{11}$ if $|y_{i,j}| \geq 1$ and is zero otherwise. Thus, the lemma follows from Proposition 3.1.6. $\square$

We shall remark that the above lemma, when applied to SL_r, yields the values of the entries of the second row of the matrix p_{r+1}.

We will write λ as $\lambda = \sum_i \lambda_i \varpi_i$ where the ϖ_i are the fundamental weights. Then the value of the character ψ_λ is given by

$$\psi_\lambda(u) = \prod_{i=1}^{r} \psi(\varpi^{-\lambda_i} x_{i,i+1}).$$

Now we come to the formula for $\psi(u)$ in terms of the w and y variables. Let

$$\Psi_\lambda^{i,j}(u) = \begin{cases} \psi\left(y_{i,j}\, \varpi^{-\lambda_i} \prod_{k=j+1}^{r+1} \frac{w_{i,k}}{w_{i+1,k}}\right) & \text{if } j = i+1 \text{ or } |y_{i+1,j}| > 1 \\ 1 & \text{otherwise.} \end{cases}$$

Proposition 3.5.2. *We have the following formula for the value of the additive character ψ on U^-, in the w_α coordinate system.*

$$\psi_\lambda(u) = \prod_{\alpha \in \Phi^+} \Psi_\lambda^\alpha(u).$$

Proof. By induction on r, it suffices to consider the case where $i = 1$. Since we know that $\psi_\lambda(u) = \prod_i \psi(x_{i,i+1}\varpi^{-\lambda_i})$, we have to keep track of the $2,1$-entry of the matrix

$e_{-\alpha_1}(x_1)\ldots e_{-\alpha_k}(x_k)p_k$ in order to express $x_{1,2}$ in terms of the w or y variables. The following sequence of events happens to this entry (note that we only need to consider $k \leq r$).

Initially, the (2,1) entry is $x_{1,2}$. For k running from $r+1$ to 3, the following happens.

First one subtracts $y_{1,k}(p_{r+1})_{2,k}$, then divides by $w_{1,k}$.

At the end of this process, one is left with y_1, and hence has a formula for x_1 in terms of the w variables. Then a simple application of the fact that ψ is an additive character, together with our knowledge of $(p_{r+1})_{2,k}$ obtained via the previous lemma produces our desired formula for $\psi_\lambda(u)$. □

With the help of the calculation of the previous lemma, as well as the characterisation of the finite crystals from Theorem 3.4.8 from the previous section, we are able to immediately deduce the following explicit description of $B(\lambda+\rho)$ in the case under consideration. The weight ρ is as usual half the sum of the positive coroots, in terms of the fundamental weights, we have $\rho = \varpi_1 + \ldots + \varpi_r$.

Proposition 3.5.3. *We have* $(\mathbf{i}, \mathbf{m}) \in B(\lambda+\rho)$ *if and only if the following inequalities hold.*

For each $(i,j) \in \Phi^+$, $m_{i,j} \geq 0$ *and*

$$\sum_{k=j}^{r+1} m_{i,k} \leq \lambda_i + 1 + \sum_{k=j}^{r} m_{i+1,k+1}. \tag{3.1}$$

In order to describe a combinatorial formula for the Whittaker function, we first decorate the tuple of integers $\mathbf{m}$ by circles and boxes. Suppose $(\mathbf{i}, \mathbf{m}) \in B(\lambda+\rho)$. For each $\alpha \in \Phi^+$, we define a weighting $w(\mathbf{m}, \alpha)$ as follows. Say that m_α is circled if $m_\alpha = 0$ and boxed if equality holds in the corresponding inequality in (3.1), ie if $\alpha = (i,j)$ and $\sum_{k=j}^{r+1} m_{i,k} = \lambda_i + 1 + \sum_{k=j+1}^{r+1} m_{i+1,k+1}$. (This vocabulary is used to match that of [10]). Corresponding to this decorating, we define a weight function

$$w(\mathbf{m},\alpha) = \begin{cases} q^{-1}g(r_\alpha, s_\alpha) & \text{if } m_\alpha \text{ is not circled} \\ 1 & \text{if } m_\alpha \text{ is circled but not boxed} \\ 0 & \text{if } m_\alpha \text{ is both boxed and circled} \end{cases} \tag{3.2}$$

where the integers r_α and s_α are defined by

$$r_{i,j} = \sum_{k \le i} m_{k,j}$$

and

$$s_{i,j} = \lambda_i + \sum_{k=j}^{r} m_{i+1,k+1} - \sum_{k=j}^{r+1} m_{i,k}.$$

Note that in the case where m_α is neither boxed nor circled, then Proposition 1.2.1 on evaluating such Gauss sums can be used to simplify the resulting expression, which only depends on the value of r_α modulo n.

Theorem 3.5.4. *The integral over $C^{\mathbf{i}}_{\mathbf{m}}$ is given by*

$$\int_{C^{\mathbf{i}}_{\mathbf{m}}} f(u)\psi_\lambda(u)du = \begin{cases} \prod_{\alpha \in \Phi^+} w(\mathbf{m}, \alpha) x_\alpha^{m_\alpha} & \text{if } (\mathbf{i}, \mathbf{m}) \in B(\lambda + \rho) \\ 0 & \text{otherwise} \end{cases}$$

Remark 3.5.5. We expect that the vanishing of the above integral over $C^{\mathbf{i}}_{\mathbf{m}}$ should occur for the case of an arbitrary root system and choice of $\mathbf{i} \in I$, whenever $(\mathbf{i}, \mathbf{m}) \notin B(\lambda + \rho)$. This would yield an expression for the metaplectic Whittaker function as a sum over $B(\lambda + \rho)$ as opposed to the $B(-\infty)$ that our expression is currently a priori a sum over.

Proof. We will first present the proof in the case where $m_\alpha > 0$ for all $\alpha \in \Phi^+$, then discuss the changes that need to be made in order to incorporate the general case.

Recall that we have the variables $w_\alpha = \varpi^{-m_\alpha} u_\alpha$. In terms of such variables, we have

$$\psi(u) = \prod_{(i,j) \in \Phi^+} \psi\left(\varpi^{s_{ij}} \frac{\prod_{k=j}^{r+1} u_{i,k}}{\prod_{k=j}^{r} u_{i+1,k+1}}\right)$$

whereas

$$f(u) = \prod_{\alpha \in \Phi^+} (q^{-\langle \rho, \alpha^\vee \rangle} x_\alpha)^{m_\alpha} (u_\alpha, \varpi)^{m_\alpha + \sum_{\alpha <_{\mathbf{i}} \beta} \langle \beta, \alpha^\vee \rangle m_\beta}.$$

We will show that the change of variables

$$t_{i,j} = \frac{\prod_{k=j}^{r+1} u_{i,k}}{\prod_{k=j}^{r} u_{i+1,k+1}}$$

transforms the integral $\int_{C^{\mathbf{i}}_{\mathbf{m}}} f(u)\psi_\lambda(u)du$ into the product of integrals over $O_F^\times$ which are the defining integrals for the Gauss sums $g(r_\alpha, s_\alpha)$, that is

$$\int_{C^{\mathbf{i}}_{\mathbf{m}}} f(u)\psi_\lambda(u)du = \prod_{\alpha\in\Phi^+} \frac{x_\alpha^{m_\alpha}}{q} \int_{O_F^\times} (t_\alpha, \varpi)^{r_\alpha}\psi(\varpi^{s_\alpha}t_\alpha)dt_\alpha.$$

In this equation, most of the appearances of the powers of q appearing in the value of f cancel out with the factors that arise from performing a change of variables. The Haar measure on $O_F^\times$ appears with the same normalisation as in Section 1.2, where the Gauss sums $g(r_\alpha, s_\alpha)$ are defined.

To see this it suffices to check that

$$\prod_{\alpha\in\Phi^+} u_\alpha^{m_\alpha+\sum_{\alpha<_{\mathbf{i}}\beta}\langle\beta,\alpha^\vee\rangle m_\beta} = \prod_{(i,j)\in\Phi^+} t_{i,j}^{r_{i,j}}.$$

A term $u_{i,j}$ on the product in the right hand side of the above product appears with a positive exponent in $t_{i,k}$ for $k \leq j$ and with a negative exponent in $t_{i-1,k}$ for $k \leq j-1$. When we pair up and cancel the occurrences from these terms as much as possible, we are left with $u_{ij}^{m_{ik}}$ from these terms. There also occurs u_{ij} with exponent $\sum_{k\leq j} m_{k,j}$ from the $t_{i,j}$ term, and an occurrence of u_{ij} with exponent $-\sum_{k\leq i-1} m_{k,i}$ from the $t_{i-1,i}$ term. Combining these gives exactly the formula we want, which is enough to prove the theorem in the case where all m_α are positive.

Now we shall consider the alterations to the above that need to be considered when some of the m_α are allowed to take the value zero.

In general, we perform the same manipulations to get a product of integrals of $(t_{i,j}, \varpi)^{r_{i,j}}$ $\psi(t_{i,j}\varpi^{s_{i,j}})$ except in two cases. The first of these is when $m_{i,j} = 0$ in which case the term $(t_{i,j}, \varpi)^{r_{i,j}}$ is no longer there, so we are left with simply an integral of an additive character over O_F, which we can easily evaluate to $w(\mathbf{m}, \alpha)$.

The more subtle case to handle is when we have $m_{i+1,j} = 0$, for then this forces the 'disappearance' of the additive character in the integrand. By Proposition 1.2.1, this is not of concern unless $s_{i,j} \leq -1$, for otherwise we have an explicit evaluation of the Gauss sums so we can see that we get our desired integral. Now note that the condition $m_{i+1,j} = 0$ implies that $s_{i,j} = s_{i,j-1} + m_{i,j-1}$. Since $m_{i,j-1} \geq 0$, we have $s_{i,j-1} \leq -1$. These inequalities imply that the integral over $t_{i,j-1}$ is zero from our previous work, unless $m_{i+1,j-1} = 0$. We can deal with this latter case because we may assume without loss of generality that

$m_{i+1,j-1} \neq 0$ (if it exists) by decreasing j if necessary.

Thus in all cases, we are led to an easily evaluated integral, yielding the desired product formula for the integral over $C^{\mathbf{i}}_{\mathbf{m}}$, so we are done. $\square$

We have thus proven the following theorem.

Theorem 3.5.6. *The value of the integral I_λ defined in (3.2) which calculates the metaplectic Whittaker function is given by*

$$I_\lambda = \sum_{(\mathbf{i},\mathbf{m})\in B(\lambda+\rho)} \prod_{\alpha\in\Phi^+} w(\mathbf{m},\alpha) x_\alpha^{m_\alpha}.$$

where the weight $w(\mathbf{m},\alpha)$ is as defined in (3.2).

Noting that the sum over the crystal $B(\lambda+\rho)$ is identical to the corresponding sum over the crystal expressing the p-part of a Weyl group multiple Dirichlet series as presented in [10], we obtain the following corollary

Corollary 3.5.7. *The coefficients of the p-part of a Weyl group multiple Dirichlet series in type A is equal to the value of a Whittaker function evaluated at a torus element on the corresponding local metaplectic group.*

Chapter 4

Hecke algebras

4.1 The spherical Hecke algebra

We call a complex valued function f on $\widetilde{G}$ anti-genuine if, for all $\zeta \in \mu_n$ and $g \in \widetilde{G}$, we have $f(\zeta g) = \zeta^{-1} f(g)$. This notion is of use to us since we are only studying genuine representations of $\widetilde{G}$. If we decompose the algebra $C_c^\infty(\widetilde{G})$ of smooth compactly supported functions on $\widetilde{G}$ into a direct sum of eigenspaces under the action of μ_n, then only the anti-genuine functions act non-trivially on a genuine representation of $\widetilde{G}$. We now define and study a version of the spherical Hecke algebra for the metaplectic group.

Considering K as a subgroup of $\widetilde{G}$ via κ^*, let $\mathcal{H}$ denote the algebra of K-bi-invariant anti-genuine compactly supported smooth (locally constant) complex valued functions. In other words, a compactly supported smooth function $f : \widetilde{G} \longrightarrow \mathbb{C}$ is in $\mathcal{H}$ if and only if $f(\zeta k_1 g k_2) = \zeta^{-1} f(g)$ for all $\zeta \in \mu_r$, $g \in G$ and $k_1, k_2 \in K$. The algebra structure is given by convolution, for $f_1, f_2 \in \mathcal{H}$, we define

$$(f_1 f_2)(g) = \int_{\widetilde{G}} f_1(h) f_2(h^{-1} g) dh$$

where the Haar measure on $\widetilde{G}$ is normalised such that $K \times \mu_n$ has measure 1.

We have the following two results about the structure of $\mathcal{H}$. In the case of $G = GL_n$, these appear in [23].

Theorem 4.1.1. *$\mathcal{H}$ is commutative.*

We will not prove this in this section, but instead note that it follows immediately from the Satake isomorphism, Theorem 4.2.1.

Theorem 4.1.2. *The support of $\mathcal{H}$ is given by $\mu_n KHK$.*

Proof. The Cartan decomposition $G = KTK$ implies that every (K, K) double coset of G contains a representative of the form ϖ^λ, and this decomposition clearly lifts to $\widetilde{G}$. So it suffices to find the set of λ for which the double coset $\mu_n K \varpi^\lambda K$ supports a function in $\mathcal{H}$.

Fix λ, and let K^λ denote the subgroup $K \cap \varpi^{-\lambda} K \varpi^\lambda$ of G. We define a function $\phi^\lambda : K^\lambda \longrightarrow \mu_n$ as follows. For $k \in K^\lambda$ there exists a unique $k' \in K$ such that $k\varpi^\lambda = \varpi^\lambda k'$. We lift this identity into $\widetilde{G}$ using our choice of splitting of K, and define $\phi^\lambda(k)$ by $k\varpi^\lambda = \phi(k)\varpi^\lambda k'$.

It is straightforward to check that ϕ^λ is a group homomorphism. Furthermore, there is a function in $\mathcal{H}$ supported on $\mu_n K \varpi^\lambda K$ if and only if the homomorphism ϕ^λ is trivial.

The normal subgroup $K_1 \cap K^\lambda$ of K^λ is a pro-p group, hence the homomorphism ϕ^λ is trivial when restricted to this subgroup.

There is a canonicial isomorphism $K^\lambda/(K_1 \cap K^\lambda) \simeq \mathbf{P}(k)$ for some parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. The above shows that ϕ^λ factors to a homomorphism from $\mathbf{P}(k)$ to μ_n. The group $\mathbf{P}(k)$ is generated by $\mathbf{T}(k)$ and unipotent elements. Since ϕ^λ is necessarily trivial on any unipotent element, it is completely determined by its restriction to $\mathbf{T}(k)$.

We know that the restriction of ϕ^λ to $\mathbf{T}(O_F)$ is trivial if and only if $\varpi^\lambda \in H$, by the definition of H. This completes our proof. □

4.2 Satake Isomorphism

The approach we shall take in presenting the Satake isomorphism was learnt by the author from a lecture of Kazhdan in the reductive case, and differs from that which is generally considered as for example in [19]. First we define a free abelian group Λ which shall be of fundamental importance throughout the rest of this paper. Let

$$\Lambda = \{\lambda \in Y \mid s(\varpi^\lambda) \in H\} = \{x \in Y \mid B(x, y) \in n\mathbb{Z} \ \forall \ y \in Y\}.$$

The equivalence of the two given presentations is a consequence of the commutator formula (1.1). This group Λ is also naturally isomorphic to the abelian group $H/(\widetilde{T} \cap K \times \mu_n)$, and carries an action of the Weyl group, inherited from the action of W on T.

The aim of this section is to prove the following.

Theorem 4.2.1 (Satake Isomorphism). *Let $\mathbb{C}[\Lambda]$ denote the group algebra of Λ. Then there is a natural isomorphism between the spherical Hecke algebra $\mathcal{H}$ and the W-invariant subalgebra, $\mathbb{C}[\Lambda]^W$.*

Let Z_Λ denote the complex affine variety $\mathrm{Hom}\,(\Lambda, \mathbb{C}^\times)$, and Γ_Λ be the ring of regular functions on Z_Λ. We shall first define a homomorphism from $\mathcal{H}$ to Γ_Λ.

To any $\chi \in Z_\Lambda$ there is an associated genuine unramified principal series representation $I(\chi) = (\pi_\chi, V_\chi)$ of $\widetilde{G}$. By Lemma 2.2.3, this representation has the property that $\dim V_\chi^K = 1$, and thus V_χ^K is a one-dimensional representation of $\mathcal{H}$. We again use $\pi_\chi : \mathcal{H} \longrightarrow \mathrm{End}\,(V_\chi^K) \cong \mathbb{C}$ to denote this representation.

From this representation, we obtain a ring homomorphism $S : \mathcal{H} \longrightarrow \Gamma_\Lambda$ given by $Sf(\chi) = \pi_\chi(f)$. This is the Satake map. A priori, the image of this map lies in the set of functions from Z_λ to $\mathbb{C}$, though it will follow from the results proven below that the image lies in the ring of regular functions on Z_λ.

For any abelian group Λ there is a canonical isomorphism between Γ_Λ and the group ring of Λ (which is actually the same as given above, if we can take $\widetilde{G} = \Lambda$ in the definition of the Satake map).

Let us identify Γ_Λ with $\mathbb{C}[\Lambda]$ via this isomorphism. Using this we will from now assume that S has image in $\mathbb{C}[\Lambda]$.

Lemma 4.2.2. *We have the following formula for the Satake map $S : \mathcal{H} \longrightarrow \mathbb{C}[\Lambda]$*

$$(Sf)(\lambda) = \delta^{1/2}(\varpi^\lambda) \int_U f(\varpi^\lambda u)du. \tag{4.1}$$

Proof. We begin by unfolding of the integral definition of the action of f on the spherical vector ϕ_K. From this we get

$$\begin{aligned}
\pi_\chi(f)\phi_K &= \int_{\widetilde{G}} f(g)\pi_\chi(g)\phi_K dg \\
&= \int_K \int_{\widetilde{B}} f(bk)\pi_\chi(bk)\phi_K d_lb\ dk \\
&= \int_{\widetilde{B}} f(b)\pi_\chi(b)\phi_K d_lb \\
&= \int_{\widetilde{T}} \int_U f(tu)\pi_\chi(tu)\phi_K du\ dt \\
&= \int_{\widetilde{T}} \left(\delta^{1/2}(t) \int_U f(tu)du\right) (\delta^{-1/2}\pi_\chi)(t)\phi_K dt.
\end{aligned}$$

It was shown in the proof of Lemma 2.2.3 that for $t \in \widetilde{T}$, $(\delta^{-1/2}\pi_\chi)(t)\phi_K = \chi(t)\phi_K$ if $t \in H$ and is zero otherwise. Thus we may restrict our integral over $\widetilde{T}$ to an integral over H. Since the integrand is invariant under $\widetilde{T} \cap K \times \mu_n$, we obtain the following sum over Λ

$$\pi_\chi(f) = \sum_{\lambda \in \Lambda} \delta^{1/2}(\varpi^\lambda) \int_U f(\varpi^\lambda u) du \chi(\varpi^\lambda).$$

Under the isomorphism $\Gamma_\Lambda \simeq \mathbb{C}[\Lambda]$, this gives us (4.1) as required. $\square$

Lemma 4.2.3. *The image of the Satake map lies in Γ_Λ^W.*

Proof. By Corollary 2.3.6, we have, for generic χ, an isomorphism between $I(\chi)^K$ and $I(\chi^w)^K$. Thus the image of the Satake map is W-invariant. To complete the proof, it remains to show that the image of S consists of regular functions on Z_λ (or equivalently that $(Sf)(\lambda)$ is nonzero for only finitely many λ). For this we use the integral expression from Lemma 4.2.2. To see this, we need to remark that any $f \in \mathcal{H}$ is compactly supported, and use [11, Proposition 4.4.4(i)]. $\square$

Theorem 4.2.4. *The Satake map S gives an isomorphism between $\mathcal{H}$ and $\mathbb{C}[\Lambda]^W$.*

Proof. For dominant $\lambda \in \Lambda$, we define basis elements c_λ and d_λ of $\mathcal{H}$ and $\mathbb{C}[\Lambda]^W$ respectively.

Let c_λ be the function in $\mathcal{H}$ that is supported on $\mu_n K \varpi^\lambda K$ and takes the value 1 at $s(\varpi^\lambda)$. That the set of all such c_λ form a basis of $\mathcal{H}$ is known from Theorem 4.1.2.

Let $d_\lambda \in \mathbb{C}[\Lambda]^W$ be the characteristic function of the orbit $W\lambda$.

Write $Sc_\lambda = \sum_\mu a_{\lambda\mu} d_\mu$. We shall show that $a_{\lambda\lambda} \neq 0$ and that $a_{\lambda\mu} = 0$ unless $\mu \leq \lambda$, which suffices to prove that S is bijective. Since we already know that S is a homomorphism, this is sufficient to prove our theorem.

To show that $a_{\lambda\lambda} \neq 0$, we must calculate $Sc_\lambda(\lambda)$. Notice that for $u \in U$, we have $\varpi^\lambda u \in K\varpi^\lambda K$ if and only if $u \in K$ so in the calculation of the integral (4.1), the integrand is non-zero only on $K \cap U$, where it takes the value 1, hence the integral is non-zero, so $a_{\lambda\lambda} \neq 0$ as desired.

To show that $a_{\lambda\mu} = 0$ unless $\mu \leq \lambda$, we again look at calculating $Sc_\lambda(\mu)$ via the integral (4.1). We again appeal to a result from the structure theory of reductive groups over local fields [11, Proposition 4.4.4(i)] to say that $\varpi^\mu U \cap K\varpi^\lambda K = 0$ unless $\mu \leq \lambda$, which immediately gives us our desired vanishing result, so we are done.

□

4.3 The dual group to a metaplectic group

Motivated by the Satake isomorphism in the previous section, we will now give a combinatorial definition of a dual group to a metaplectic group. This group $\widetilde{G}^\vee$ will be a split reductive group, so to define it, it will suffice to give a root datum (X, Φ, X', Φ').

We use Δ to denote the set of all coroots. Throughout this section, lower case Greek letters will be used to denote coroots. If α is a simple coroot, recall that the integer n_α is defined to be the quotient $n_\alpha = \frac{n}{(n, Q(\alpha))}$ (where $(\cdot,\cdot)$ here is used to denote the greatest common divisor).

We define a root datum (X, Φ, X', Φ') by

$$\begin{aligned} X &= \Lambda, \\ \Phi &= \{n_\alpha \alpha \mid \alpha \in \Delta\}, \\ X' &= \mathrm{Hom}\,(\Lambda, \mathbb{Z}) \subset \mathrm{Hom}\,(\mathbf{T}, \mathbb{G}_m) \otimes \mathbb{Q}, \\ \Phi' &= \{n_\alpha^{-1} \alpha^\vee \mid \alpha \in \Delta\}, \end{aligned}$$

and we define the dual group $\widetilde{G}^\vee$ of $\widetilde{G}$ to be the reductive group associated to this root datum.

Theorem 4.3.1. *The quadruple (X, Φ, X', Φ') defines a root datum.*

Proof. To check that Φ and Φ' are stable under the Weyl group is straightforward. For example, if $w\alpha = \beta$, then $Q(\alpha) = Q(\beta)$ so $wn_\alpha\alpha = n_\beta\beta$. The only part involving significant work is to check that $\Phi \subset X$ and $\Phi' \subset X'$.

To check that $\Phi \subset X$, it suffices to show that for all $\alpha \in \Delta$ and $y \in Y$ we have that $B(\alpha, y)$ is divisible by $Q(\alpha)$.

Consider the set $L_y = y + \mathbb{Z}\alpha$. It is a w_α stable subset of Y. There are two possibilities, either L_y contains z which is fixed by w_α or L_α contains z such that $w_\alpha z = z + \alpha$.

In the former case, consider the $\mathbb{Q}$-subspace of $Y \otimes \mathbb{Q}$ spanned by z and α. On this subspace we have $Q(m\alpha + nz) = Am^2 + Bn^2 + Cmn$ for some $A, B, C \in \mathbb{Q}$. Since Q is invariant under w_α, we must have that $C = 0$. Then $B(\alpha, z) = 0$, so since $Q(\alpha)$ divides $B(\alpha, \alpha)$, it must divide $B(\alpha, y)$.

In the latter case, we calculate that $B(\alpha, z) = -Q(\alpha)$, so proceed as in the former case, so we are done.

We now show that $\Phi' \subset X'$.

Firstly we use the fact that $B(\alpha, \alpha) = 2Q(\alpha)$ to conclude that

$$n_\alpha \mathbb{Z}\alpha \subset X \cap \mathbb{Q}\alpha \subset \frac{n_\alpha}{2}\mathbb{Z}\alpha.$$

Now consider some $\beta \in X$. and let $M_\beta = (\beta + \mathbb{Q}\alpha) \cap X$. A priori, there are three options.

The first is that there exists $\gamma \in M_\beta$ such that $w_\alpha \gamma = \gamma$, which implies $\langle \alpha^\vee, \gamma \rangle = 0$.

The second is that there exists $\gamma \in M_\beta$ such that $w_\alpha \gamma = \gamma + n_\alpha \alpha$ which implies $\langle \alpha^\vee, \gamma \rangle = n_\alpha$.

In the third potential case we would have $\gamma \in M_\beta$ such that $w_\alpha \gamma = \gamma + \frac{n_\alpha}{2}\alpha$. For this to occur, we would require that $2 | n_\alpha$, so in this case $B(\gamma, \alpha) \notin n\mathbb{Z}$. However this last statement implies that $\gamma \notin X$, which cannot occur.

Thus since we know that $\beta = \gamma + \frac{kn_\alpha}{2}\alpha$ for some integer k, we obtain that $\langle \beta, \alpha^\vee \rangle \in n_\alpha \mathbb{Z}$. This shows that $n_\alpha^{-1}\alpha^\vee \in \mathrm{Hom}(X, \mathbb{Z}) = X'$, as required.

□

Thus we have a root datum, so defining $\widetilde{G}^\vee$ as the split reductive group corresponding to this root datum is well-defined.

As a consequence, we may consider the Satake isomorphism to be the existence of a natural isomorphism

$$\mathcal{H} \cong \mathbb{C}[\Lambda]^W \cong K_0(\mathrm{Rep}(\widetilde{G}^\vee)) \otimes \mathbb{C}.$$

4.4 Iwahori Hecke Algebra

There is an alternative Hecke algebra associated to the group $\widetilde{G}$, defined in the same fashion as the spherical Hecke algebra $\mathcal{H}$, but considering a standard Iwahori subgroup I (defined to be the inverse image of $\mathbf{B}(k)$ under the surjection $K \to \mathbf{G}(k)$) in place of the hyperspecial maximal compact subgroup K. We will denote this Hecke algebra by $\mathcal{H}(\widetilde{G}, I)$, it is the algebra of antigenuine I-biinvariant compactly supported locally constant functions on $\widetilde{G}$.

Let J denote the normaliser in $\widetilde{G}$ of $\widetilde{T} \cap K$.

Theorem 4.4.1. *The support of the algebra $\mathcal{H}(\widetilde{G}, I)$ is IJI.*

Proof. We also use the decomposition $G = IMI$. Suppose that $t \in M$ and $t \notin J$. Then there exists $k \in \widetilde{T} \cap K$ such that $tkt^{-1} \notin \widetilde{T} \cap K$. Since we are assuming $t \in WT$, we have that $p(t) \in T \cap K$. Thus $tkt^{-1} = \zeta k'$ for some $k' \in \widetilde{T} \cap K$ and $\zeta \in \mu_n$ with $\zeta \neq 1$. Hence any $f \in \mathcal{H}(\widetilde{G}, I)$ has $f(t) = 0$ so we have proved that the support of $\mathcal{H}(\widetilde{G}, I)$ lies in IJI.

For the reverse implication we need to show that if $t \in J$ then there exists $f \in \mathcal{H}(\widetilde{G}, I)$ with $f(t) \neq 0$. To do this we need to show that whenever $p(i_1 t i_2) = p(t)$ for $i_1, i_2 \in I$, then $i_1 t i_2 = t$.

Let I_p denote the maximal pro-p subgroup of I (it is the inverse image of $\mathbf{U}(k)$ under the projection $K \to \mathbf{G}(k)$). The torus $\mathbf{T}(k)$ over the residue field lifts to I and every element of I can be uniquely written as a product of an element of $\mathbf{T}(k)$ with an element of I_p.

After projection to G, $i_2 \in I \cap t^{-1}It$. Write $i_2 = j_1 j_2$ with $j_1 \in \mathbf{T}(k)$ and $j_2 \in I_p$. Then $ti_2t^{-1} = tj_1t^{-1}tj_2t^{-1}$. We have $tj_1t^{-1} \in \mathbf{T}(k)$ because t normalises $\widetilde{T} \cap K$ and $\mathbf{T}(k)$ consists of all elements of order $q-1$ in this group. Since tj_2t^{-1} topologically generates a pro-p group, it must be that $tj_2t^{-1} \in I_p$ since it is a priori in $\tilde{I}$ which also has a unique maximal pro-p subgroup. Thus $ti_2t^{-1} = tj_1t^{-1}tj_2t^{-1} \in I$ so we are done. □

Let W_a denote inverse image of Λ under the projection from the affine Weyl group to Q. Then there is an isomorphism $I \backslash IJI/I \simeq W_a$. As a corollary of the above theorem, we are able to exhibit a basis for $\mathcal{H}(\widetilde{G}, I)$. For any $w \in W$, we are able to exhibit a choice of a lifting $w \in \widetilde{G}$ which is an element of our embedding $W_0 \hookrightarrow \widetilde{G}$ from the discussion at the end of section 1.6. There is an embedding $\Lambda \subset W_a$ and $W \subset W_a$. Using these inclusions, we identify elements of λ as elements of W_a and for each simple coroot α denote by $s_\alpha \in W_a$ the corresponding simple reflection.

Corollary 4.4.2. *For each $w \in W_a$, then there is a function T_w in $\mathcal{H}(\widetilde{G}, I)$, supported on $\mu_n IwI$ and taking the value 1 at w. Then the collection of these T_w for $w \in W_a$, forms a $\mathbb{C}$ basis for the algebra $\mathcal{H}(\widetilde{G}, I)$.*

It is possible to write down a system of generators and relations for the algebra $\mathcal{H}(\widetilde{G}, I)$. The following is a corrected version of [38, Proposition 3.1.2]. The change is in the definition of Savin's integer m, which has been replaced by n_α (although $m = n_\alpha$ in a large number of cases, in general they are not even equal in the rank one case).

Let Λ^+ denote the set of dominant elements of Λ and Δ denote the set of simple coroots.

Theorem 4.4.3. *The following relations hold in $\mathcal{H}(\widetilde{G}, I)$.*

1. *$T_\lambda T_\mu = T_{\lambda+\mu}$ for $\lambda, \mu \in \Lambda^+$.*
2. *If $s_\alpha \lambda = \lambda$ for $\alpha \in \Delta$ and $\lambda \in \Lambda^+$ then T_{s_α} and T_λ commute.*
3. *If $\lambda \in \Lambda^+$ and $\langle \alpha^\vee, \lambda \rangle = n_\alpha$ then*
$$T_\lambda T_{s_\alpha}^{-1} T_\lambda T_{s_\alpha}^{-1} = q^{n_\alpha - 1} T_{2\lambda - n_\alpha \alpha}.$$
4. *If $\lambda \in \Lambda^+$ and $\langle \alpha, \lambda \rangle = 2n_\alpha$ then*
$$T_\lambda T_{s_\alpha}^{-1} T_\lambda T_{s_\alpha}^{-1} = q^{2n_\alpha - 1} T_{2\lambda - 2n_\alpha \alpha} + (q-1) q^{n_\alpha - 1} T_{2\lambda - n_\alpha \alpha} T_{s_\alpha}^{-1}.$$
5. *$(T_{s_\alpha} - q)(T_{s_\alpha} + 1) = 0$ for $\alpha \in \Delta$.*
6. *For $w_1, w_2 \in W$ with $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ we have $T_{w_1 w_2} = T_{w_1} T_{w_2}$.*

Proof. The proof given by Savin [38] is applicable here and correct until we reduce to a rank one calculation in proving parts 3 and 4. We will present this rank one calculation here. It does not appear in [38] and is the source of the inaccurate statement in [38, Proposition 3.1.2]. To carry out this computation, we will be making use of the explicit formulae for the 2-cocyle and the splitting given in equations (1.6) and (1.1) respectively.

In the rank one case, the statements of parts 3 and 4 simplify to the following, where s is the sole reflection in the Weyl group.

(3') If $\lambda = n_\alpha \alpha / 2 \in \Lambda^+$ then
$$T_\lambda T_s^{-1} T_\lambda = q^{n_\alpha - 1} T_s$$

(4') If $\lambda = n_\alpha \alpha \in \Lambda^+$ then

$$T_\lambda T_s^{-1} T_\lambda = q^{2n_\alpha - 1} T_s + (q-1) q^{n_\alpha - 1} T_\lambda.$$

We know from Savin's proof that $T_{s_i}^{-1} T_\lambda = T_{s_i \lambda}$ and so need to calculate the product $T_\lambda T_{s\lambda}$. In particular, we need to calculate $T_\lambda T_{s\lambda}(\varpi^\lambda)$, which is the task we shall accomplish.

Let us first consider the case where our rank one group is $G = SL_2$. We may thus write $\varpi^\lambda = \left(\begin{smallmatrix} \varpi^l & 0 \\ 0 & \varpi^{-l} \end{smallmatrix}\right)$ for some integer l (actually $l = \langle \alpha^\vee, \lambda \rangle / 2$). Note that $2lQ(\alpha)$ is divisible by n, which will have the consequence that all powers of Hilbert symbols that appear will be ± 1, a feature we will exploit, simplifying our expressions by freely inverting such symbols on a whim.

We write $T_\lambda T_{s\lambda}(\varpi^\lambda)$ as an integral over $\widetilde{G}/\mu_n \simeq G$.

$$T_\lambda T_{s\lambda}(\varpi^\lambda) = \int_G T_\lambda(h) T_{s\lambda}(h^{-1}\varpi^\lambda) dh.$$

This integrand is non-zero when $h^{-1} \in I\varpi^{-\lambda} I \cap I s \varpi^{-\lambda} I \varpi^{-\lambda}$. We shall work modulo I on the left. Thus we have

$$h^{-1} = \begin{pmatrix} 0 & -\varpi^{-l} \\ \varpi^l & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \varpi^{-l} & 0 \\ 0 & \varpi^l \end{pmatrix}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in I$. The condition for $h^{-1} \in I\varpi^{-\lambda} I$ is equivalent to $c = \varpi^l u$ for some $u \in O_F^\times$.

We calculate $h = \left(\begin{smallmatrix} b\varpi^{2l} & d \\ -a & -u\varpi^{-l} \end{smallmatrix}\right)$ and $s(h)^{-1} = s(h^{-1})$.

For $i_1 = \left(\begin{smallmatrix} -u & -d\varpi^l \\ 0 & -u^{-1} \end{smallmatrix}\right)$ and $i_2 = \left(\begin{smallmatrix} 1 & 0 \\ -au^{-1}\varpi^l & 1 \end{smallmatrix}\right)$ we have $i_1 h i_2 = \varpi^\lambda$, $\kappa(i_1) = \kappa(i_2) = 1$ and $\sigma(i_1 h, i_2)\sigma(i_1, h) = (au, \varpi^{lQ(\alpha)})$.

For $i_3 = \left(\begin{smallmatrix} d & -b \\ -u\varpi^l & a \end{smallmatrix}\right)$, we have $h^{-1}\varpi^\lambda i_3 = s\varpi^\lambda$, $\kappa(i_3) = (a, \varpi^\lambda)$ and $\sigma(h^{-1}, \varpi^\lambda)\sigma(h^{-1}\varpi^\lambda, i_3) = (a, \varpi^{lQ(\alpha)})$.

Thus overall our integrand $T_\lambda(h) T_{s\lambda}(h^{-1}\varpi^\lambda)$ is equal to $(au, \varpi^{lQ(\alpha)})$ where it is supported. Hence the integral $T_\lambda T_{s\lambda}(\varpi^\lambda)$ is equal zero if n does not divide $lQ(\alpha)$ and the value of the appropriate volume, namely $q^{2n_\alpha - 1}$, otherwise. To complete the proof in the SL_2 case, we need to note that in case 3, $2l = n_\alpha$ and thus n does not divide $lQ(\alpha)$ as can be seen by looking at 2-adic valuations. In case 4, $l = n_\alpha$ so n trivially divides $lQ(\alpha)$.

Now we turn to the case of $G = PGL_2$. We write $\varpi^\lambda = \left(\begin{smallmatrix} \varpi^l & 0 \\ 0 & 1 \end{smallmatrix}\right)$ In order to have our integrand $T_\lambda(h)T_{s\lambda}(h^{-1}\varpi^\lambda)$ non-zero, by consideration of the valuation of the determinant, we must have that l is even. This immediately proves our result when l is odd. For in case 4, for PGL_2 we have $l = 2n_\alpha$, so if l is odd, we must be in case 3.

If $\lambda \in \Lambda^+$ is such that $\langle \alpha^\vee, \lambda \rangle = 2n_\alpha$ and $\lambda/2 \in \Lambda^+$, then the equation (4) is a formal consequence of (3) and (5). Thus we may reduce to the case where λ is a minimal non-zero element of Λ^+. This implies that l divides n, so in particular, n is even.

Let E be an unramified quadratic extension of F. Consider the natural map $SL_2(E) \to PGL_2(E)$ and restrict this to the preimage of $PGL_2(F)$. Note that ϖ^λ and all elements of the Iwahori subgroup I lie in the image of this map. Accordingly, we will be able to make use of the above calculation for $SL_2(E)$.

Since n is even, $\frac{q+1}{2} \equiv 1 \pmod{n}$. For $s, t \in E$ with $s^2, t^2 \in F$, we thus have the following identity of Hilbert symbols.

$$(s^2, t^2)_F^{Q(\alpha/2)} = \overline{\left((-1)^{v(s)v(t)} \frac{s^{v(t)}}{t^{v(s)}}\right)}^{\frac{q-1}{n} Q(\alpha)} = \overline{\left((-1)^{v(s)v(t)} \frac{s^{v(t)}}{t^{v(s)}}\right)}^{\frac{q^2-1}{n} \frac{Q(\alpha)}{2}} = (s,t)_E^{Q(\alpha)/2}.$$

We interpret this in the following manner. Since our central extensions are determined by their restriction to maximal tori, this shows that the pullback of the extension of $PGL_2(F)$ to its inverse image in $SL_2(E)$ is the same as the restriction of the central extension on $SL_2(E)$ corresponding to the quadratic form Q' defined by $Q' = Q/2$. We are able to push forward a cocycle on SL_2 to a cocycle on PGL_2 since the centre of SL_2 remains central when lifted to $\widetilde{SL_2}$.

As a result of this relationship between the covers of $PGL_2(F)$ and $SL_2(E)$, we are able to use the SL_2 calculations above for the proof in the PGL_2 case. We have $\langle \alpha^\vee, \lambda \rangle = l$ and $T_\lambda(h)T_{s\lambda}(h^{-1}\varpi^\lambda)$ is non-zero if and only if n divides $lQ(\alpha)/2$.

In case 3, $l = n_\alpha$. Since n is known to be even, it does not divide $lQ(\alpha)/2$ by the same 2-adic argument as in the SL_2 case.

In case 4, $l = 2n_\alpha$ and in this case n trivially divides $lQ(\alpha)/2$.

This completes our calculation, and so combined with the work in [38], completes the proof. □

There is a stronger statement, giving a presentation for the Hecke algebra $\mathcal{H}(\widetilde{G}, I)$.

Theorem 4.4.4. *[38] The set of relations presented in Theorem 4.4.3 provides a complete* *set of relations for the algebra* $\mathcal{H}(\widetilde{G}, I)$.

Proof. The proof of Savin [38] of this theorem goes through without change. $\square$

4.5 Further work with the dual group

Corollary 4.5.1 ([38]). *Suppose that* $\widetilde{G}$ *and* $\widetilde{H}$ *are two metaplectic groups with isomorphic dual groups* $\widetilde{G}^\vee \cong \widetilde{H}^\vee$ *and Iwahori subgroups* $I^{\widetilde{G}}$ *and* $I^{\widetilde{H}}$ *respectively. Then there is an isomorphism of Iwahori-Hecke algebras*

$$\mathcal{H}_\epsilon(\widetilde{G}, I^{\widetilde{G}}) \cong \mathcal{H}_\epsilon(\widetilde{H}, I^{\widetilde{H}}).$$

Proof. This is an immediate consequence of the description of these Hecke algebras in terms of generators and relations in Theorem 4.4.4. To see this explicitly, we rewrite the relations without any occurrences of n_α in the exponents of q by defining new variables $U_s = T_s$ and $U_\lambda = q^{-\langle \rho^\vee, \lambda \rangle} T_\lambda$. $\square$

To the data of a metaplectic cover of a split group (that is the group G, the quadratic form Q and the degree of the cover n), let us propose to define the L-group of $\widetilde{G}$ to be the complex reductive group $\widetilde{G}^\vee(\mathbb{C})$. We hope that this definition will provide a way to bring the study of the metaplectic groups into the paradigm that is the Langlands functoriality conjectures.

The above corollary together with the metaplectic Satake isomorphism provide a starting point for correspondences between local representations with an Iwahori-fixed or spherical vector respectively. In the spherical case, we have the following.

Proposition 4.5.2. *Suppose* $\widetilde{G}$ *and* $\widetilde{H}$ *are two metaplectic (possibly reductive) groups with a continuous homomorphism* ${}^L\widetilde{G} \to {}^L\widetilde{H}$. *Then there is a natural correspondence from spherical representations of* $\widetilde{G}$ *to spherical representations of* $\widetilde{H}$.

Proof. The homomorphism ${}^L\widetilde{G} \to {}^L\widetilde{H}$ defines a functor $\mathrm{Rep}({}^L H) \to \mathrm{Rep}({}^L G)$. Taking Grothendieck groups and using the Satake isomorphism we obtain a natural morphism of spherical Hecke algebras $\mathcal{H}(\widetilde{H}, K) \to \mathcal{H}(\widetilde{G}, K)$, hence a map between representations of these spherical Hecke algebras, and thus a correspondence of representations from spherical representations of $\widetilde{G}$ to spherical representations of $\widetilde{H}$. $\square$

4.6 A Geometric Approach

We now provide a short discussion of a categorified version of the metaplectic Satake isomorphism due to Finkelberg and Lysenko [18]. Suppose that F is a field of Laurent series $F = k((t))$ over a field k with some mild assumption on the characteristic of k not being too small. Corresponding to $\widetilde{G}$, there is a central extension of the loop group $\mathbf{G}(F)$ by $\mathbb{G}_m(k)$ as group ind-schemes over k. This central extension splits over $\mathbf{G}(O_F)$, so we obtain a $\mathbb{G}_m$ torsor over the affine Grassmannian $Gr = \mathbf{G}(F)/\mathbf{G}(O_F)$ (as an ind-scheme over k). The group $K = \mathbf{G}(O_F)$ acts on the total space of this torsor E° by left multiplication and the group μ_n acts by multiplication fibrewise. Again choose a faithful character ϵ of $\mu_n(k)$. Consider ϵ as a representation of $\pi_1(\mathbb{G}_m)$ and let L^ϵ be the corresponding one dimensional local system on $\mathbb{G}_m$. One considers the category of perverse sheaves on E° which are K and $(\mathbb{G}_m, L^\epsilon)$-equivariant. Finkelberg and Lysenko give this category the structure of a tensor category and show that it is equivalent to the category of representations of a reductive algebraic group. They construct explicitly the root system of this group and it can be seen to be the same as the root system constructed above for the group we denoted $\widetilde{G}^\vee$.

We briefly discuss some of this theory. A generalisation which will take us too far afield is developed in the thesis of Reich [36], although the language of gerbes that is used in this generalisation is perhaps more natural. Let X denote the affine Grassmannian. Throughout, λ will be assumed to be a dominant coweight.

There is a $\mathbb{G}_m$ action on X, namely that $z \in \mathbb{G}_m$ acts on X by $t \mapsto zt$. The points t^λ for $\lambda \in \Lambda$ comprise all fixed points of this action.

There is a fibration $\pi_\lambda : Kt^\lambda \longrightarrow G(k)t^\lambda$ given by reduction modulo t in K. The stabiliser of t^λ under the action of $G(k)$ is $P^\lambda(k)$ for some parabolic subgroup P^λ associated to λ. π_λ realises Kt^λ as a vector bundle over G/P^λ. Hence the corresponding map of Picard groups $\pi_\lambda^* : \mathrm{Pic}\,(G/P^\lambda) \longrightarrow \mathrm{Pic}\,(Kt^\lambda)$ is an isomorphism.

Let E be the line bundle over X corresponding to the $\mathbb{G}_m$-torsor E°. The torus T lies in the stabiliser of t^λ. So we have a morphism of algebraic groups $T \to GL_1$. We now determine this map.

In coordinates it sends $(t_1, \ldots, t_r) \mapsto \prod_i t_i^{\sum_j \lambda_j b_{ij}}$ (obtained using the commutator formula (1.1) for the central extension). This determines a linear map $i_Q : Y(T) \longrightarrow X(T)$ for which

$$E|_{Kt^{\lambda}} = \pi_{\lambda}^{*}(O(i_Q(\lambda))).$$

We have $\langle i_Q(\lambda), \mu\rangle = B(\lambda, \mu)$.

Let K_λ be the stabiliser of t^λ under the K action. Then $K_\lambda \to GL_1$ factors through P_λ (note that in the function case, showing this was a little harder than originally anticipated). Thus to check some sheaf is equivariant, it suffices to check T equivariance, since T maps surjectively onto $P^\lambda/[P^\lambda, P^\lambda]$.

For Kt^λ to support a K and $(\widetilde{G}_m, L^\zeta)$-equivariant sheaf, we must have that $i_Q(\lambda) \in nX(T)$ by checking T-equivariance. Since $i_Q(\lambda) \in nX(T)$ if and only if $n|B(\lambda, \mu)$ for all μ, this condition is equivalent to $\lambda \in \Lambda$.

Suppose $\lambda \in \Lambda$ and write E for its restriction to Kt^λ. Then $E \cong L^{\otimes n}$ for some line bundle L on Kt^λ. Removing the zero section of these line bundles, there is a map of total punctured spaces $L^\circ \to E^\circ$ which fibrewise is given by $z \mapsto z^n$. This map, together with the character ϵ determines a K and $(\mathbb{G}_m, L^\epsilon)$ equivariant local system A_λ on E°. The corresponding simple element of our category of perverse sheaves will be the intermediate extension of this local system.

This gives a complete description of the simple objects of the category of K and $(\mathbb{G}_m, L^\epsilon)$ equivariant perverse sheaves on E°. The tensor structure on this category is more complicated to construct, and so we do not include it in our brief summary, the interested reader can look at the works of [18] and [33] for this tensor structure in the twisted and classical cases respectively.

Appendix A

Some Global Theory

The argument in this appendix was worked out in conjunction with Solomon Friedberg.

In this appendix, we relate the Whittaker functions considered in the main body of this work with Fourier coefficients of Eisenstein series on an adelic version of the metaplectic group, and show how one obtains twisted multiplicativity for multiple Dirichlet series.

For this section only, we take a break from our usual practice and now let F be a global field, again containing $2n$ $2n$-th roots of unity, while $\mathbf{G}$ will be a split reductive group over F. Write $\mathbb{A}_F$ for the adele ring of F. There is a central extension $\widetilde{G}(\mathbb{A}_F)$ of the adelic group $\mathbf{G}(\mathbb{A}_F)$ by the group μ_n [12, §10]. At each finite place v of F, this extension restricts to the local extension of $\mathbf{G}(F_v)$ considered previously. This central extension has the important property that the subgroup $\mathbf{G}(F)$, embedded diagonally into $\mathbf{G}(\mathbb{A}_F)$ is split under the extension. The reason for this essentially comes down to the reciprocity law

$$\prod_v (x,y)_v = 1$$

for all $x, y \in F^\times$, where the product is taken over all places v of F.

Unlike in the non-metaplectic case, there is no canonical decomposition $g = (g_v)_v$ of an element $g \in \widetilde{G}(F)$, nor is there a canonical projection map from $\widetilde{G}(\mathbb{A}_F)$ to $\widetilde{G}(F_v)$. However we are able to circumvent these difficulties by choosing an explicit cocycle $\sigma \in H^2(\mathbf{G}(\mathbb{A}_F), \mu_n)$ representing the central extension $\widetilde{G}(\mathbb{A}_F)$. For each place v we have a local cocycle $\sigma_v \in Z^2(\mathbf{G}(F_v), \mu_n)$ with the property that σ_v is cohomologically trivial when restricted to the maximal compact subgroup $K_v = \mathbf{G}(O_v)$ for all but a finite number of places v. For each of these places, we have a function $\kappa_v = : K_v \longrightarrow \mu_n$ which we may extend in any continuous manner to a μ_n-valued function on $\mathbf{G}(F_v)$, trivial on $\mathbf{T}(F_v)$. We now define a new cocycle σ'_v, which is cohomologous to σ_v by

$$\sigma'_v(g,h) = \frac{\sigma_v(g,h)\kappa_v(gh)}{\kappa_v(g)\kappa_v(h)}.$$

This cocycle has the important property that it takes the value 1 when restricted to $K_v \times K_v$.

Now, for two elements $g = (g_v)_v$ and $h = (h_v)_v$ of $\mathbf{G}(A_F)$, we define the cocycle σ by

$$\sigma(g,h) = \prod_v \sigma'_v(g_v, h_v).$$

Using this cocycle σ, we are able to identify an element of $\widetilde{G}(A_F)$ with an ordered pair (g, ζ) with $g \in \mathbf{G}(A_F)$ and $\zeta \in \mu_n$. In this manner, for an element $g \in \widetilde{G}(\mathbb{A}_F)$ we are able to speak of its local component g_v, and define a map $\widetilde{G}(A_F) \to \widetilde{G}(F_v)$ by $((g_v)_v, \zeta) \mapsto (g_v, \zeta)$. The following manipulations require that we have this decomposition and projection map at our disposal.

Following [34, II.1.5], let $f = \otimes_v f_v$ be an automorphic function on $\mathbf{U}(\mathbb{A}_F)\mathbf{T}(k)\backslash\widetilde{G}(\mathbb{A}_F)$ that lies in the π-isotypic component for some irreducible automorphic representation π of T. This implies that f_v will be the spherical vector in a principal series representation normalised so that $f_v(1) = 1$ for almost all places v. The Eisenstein series $E(f, g)$ is defined by the sum

$$E(f,g) = \sum_{\gamma \in B(F)\backslash G(F)} f(\gamma g).$$

Let $\psi = \prod_v \psi_v$ be a principal character of $U(\mathbb{A}_F)$. The global Fourier-Whittaker coefficient of this Eisenstein series with respect to ψ is given by the integral

$$W(g) = \int_{U(F)\backslash U(\mathbb{A}_F)} E(f, w_0 u g)\psi(u)du.$$

We are able to consider $\mathbf{U}(\mathbb{A}_F)$ as a subgroup of $\widetilde{G}(\mathbb{A}_F)$ since it is split canonically in our central extension [34, Appendix 1].

We write this as a product of local integrals via an 'unfolding' technique common in the theory of automorphic forms. When expanding E as a sum over $\gamma \in \mathbf{B}(F)\backslash\mathbf{G}(F)$, only terms in the largest Bruhat cell Bw_0Uw_0 will give a non-zero contribution to the Fourier coefficient. We obtain

$$\begin{aligned} W(g) &= \int_{U(F)\backslash U(\mathbb{A}_F)} \sum_{\gamma\in B(F)\backslash G(F)} f(\gamma w_0 u g)\psi(u)du \\ &= \int_{U(F)\backslash U(\mathbb{A}_F)} \sum_{n\in U(F)} f(w_0 n w_0 w_0 u g)\psi(u)du \\ &= \int_{U(\mathbb{A}_F)} f(w_0 u g)\psi(u)du \\ &= \prod_v \int_{U(F_v)} f_v(w_0 u_v g_v)\psi_v(u_v)du_v. \end{aligned}$$

Now let us choose g to be an element of $\mathbf{T}(F)$, embedded in the canonical way in $\widetilde{G}(\mathbb{A}_F)$. Write s_v for the composition of $\mathbf{G}(F) \to \widetilde{G}(\mathbb{A}_F) \to \widetilde{G}(F_v)$ (the reader should beware that this last map is not a homomorphism). Then we have

$$W(g) = \prod_v W_v(s_v(g)).$$

It is the act of choosing a uniformiser ϖ_v at each finite place, and writing $s_v(g) = p^\lambda x$ for some unit x and cocharacter λ that introduces the Hilbert symbols that appear in the phenomenon of twisted multiplicativity for multiple Dirichlet series.

Let us choose $t \in \mathbf{T}(F)$ and write it as as $t = \prod_v t_v$, where t_v is a power of the uniformiser ϖ_v at each (good) place. Then we have

$$\begin{aligned} W(t) &= \prod_v W_v(s_v(t)) \\ &= \prod_v W_v\left(s_v(t_v)s_v(\prod_{w\neq v} t_w)\sigma_v(t_v, \prod_{w\neq v} t_w)\right) \\ &= \prod_v W_v(t_v)\sigma_v(t_v, \prod_{w\neq v} t_w). \end{aligned}$$

We are free to use the local cocycle σ_v instead of σ'_v since we are only evaluating our cocycle at torus elements.

The following theorem is the main result of this appendix, which by this stage we have largely proved. The notation $\left(\frac{\cdot}{\cdot}\right)$ denotes the global power residue symbol.

Theorem A.0.1 (Twisted Multiplicativity). *For $s, t \in bfT(F)$, the corresponding global*

Whittaker coefficients satisfy the following twisted multiplicativity relation

$$W(st)W(1) = W(s)W(t)\prod_{i\leq j}\left(\frac{t_j}{s_i}\right)^{q_{ij}}\left(\frac{s_j}{t_i}\right)^{q_{ij}}$$

where the notation is the same as was introduced in (1.1).

Remark A.0.2. If we consider the case where $\mathbf{G}$ is simple and simply connected and Q is the quadratic form taking the value 1 on a short coroot, then the factor of the root of unity which we see in the above equation is exactly the factor $\xi(s,t)$ from equation 3.39 of [16].

Proof. The remaining ingredients that we need to complete the proof is the relation between the global power residue symbol and the local Hilbert symbols, namely that

$$\left(\frac{a}{b}\right) = \prod_v (a, b_v)_v$$

for coprime a and $b = \prod_v b_v$, together with the multiplicative properties of the Hilbert symbol and the explicit formula for the cocyle (1.1). Note that we need to use our assumption that $\mu_{2n} \subset F$ here to write the product over $w \neq v$ as a product over all places w, using the identity $(t,t)_v = 1$. $\square$

Appendix B

Connection with Kamnitzer's MV polytopes

The purpose of this appendix is to make precise the connection between our decomposition of U^- into cells $C_{\mathbf{m}}^{\mathbf{i}}$ and Kamnitzer's decomposition of the affine Grassmannian as a disjoint union [20, Proposition 4.1]. To avoid introducing a swath of notation for this small section, we will freely use the notation of [20] and the reader may find any relevant definitions there. Before we begin, we give a word of caution. Namely, Kamnitzer defines the affine Grassmannian as the left quotient $K\backslash G$ whereas we have been working with the right quotient G/K.

Our objective in this appendix is to prove the following. The sets $A^{\mathbf{i}}(\mathbf{m})$ are defined in [20, §4].

Proposition B.0.1. *Write ϕ_0 for the projection from G to the affine Grassmannian. Then we have the equality $A^{\mathbf{i}}(\mathbf{m}) = \phi_0(C_{\mathbf{m}}^{\mathbf{i}})$.*

Proof. For $[u] = L \in \cap_{w\in W} S_w^{\mu_w}$, we have $P(L) = \cap_{w\in W} C_w^{\mu_w}$. We care about the sequence of vertices $\mu_{w_1}, \mu_{w_2}, \ldots, \mu_{w_N}$.

Clearly $\mu_{w_0} = \mu_e = 0$. To find the other μ_{w_i}, we need to calculate the Iwasawa decomposition $w^{-1}uw = kb$ for $u \in U$.

Write $u = e_{\alpha_1}(x_1)e_{\alpha_2}(x_2)\ldots e_{\alpha_N}(x_N)$ for our ordering $\alpha_1 <_{\mathbf{i}} \cdots <_{\mathbf{i}} \alpha_N$ of the positive roots, and $x_i \in F$.

Then $s_1^{-1}us_1 = e_{-\alpha_1}(x_1)s_1u's_1$. Note that $s_1^{-1}u's_1 \in U$. We perform a rank one Iwasawa decomposition on $e_{-\alpha_1}(x_1)$ calculating μ_{w_1} as desired.

So we have $w_1^{-1}uw_1 = k_1b_1w_1^{-1}u'w_1$. To continue the calculation, we have to push the b_1 to the right, as in Algorithm 3.1.4.

Now we conjugate by s_2. Note that K is invariant under s_2. We still have $s_2^{-1}b_1's_2 \in B$, though its diagonal part will have changed. $s_2^{-1}u_1's_2$ will have a leading term, which we apply the rank one Iwasawa decomposition to, to calculate μ_2 as desired. We continue this process inductively.

Notice that the calculations required to calculate these μ_{w_i} are exactly the same as in the Iwasawa decomposition algorithm, Algorithm 3.1.4. Hence we may identify their outputs, and thus $A^{\mathbf{i}}(\mathbf{m}) = C^{\mathbf{i}}_{\mathbf{m}}$ as requried. $\square$

Bibliography

[1] Jared E. Anderson. A polytope calculus for semisimple groups. *Duke Math. J.*, 116(3):567–588, 2003.

[2] E. Arbarello, C. De Concini, and V. G. Kac. The infinite wedge representation and the reciprocity law for algebraic curves. In *Theta functions—Bowdoin 1987, Part 1 (Brunswick, ME, 1987)*, volume 49 of *Proc. Sympos. Pure Math.*, pages 171–190. Amer. Math. Soc., Providence, RI, 1989.

[3] Pierre Baumann and Stéphane Gaussent. On Mirković-Vilonen cycles and crystal combinatorics. *Represent. Theory*, 12:83–130, 2008.

[4] Arkady Berenstein and Andrei Zelevinsky. Tensor product multiplicities, canonical bases and totally positive varieties. *Invent. Math.*, 143(1):77–128, 2001.

[5] I. N. Bernstein and A. V. Zelevinsky. Induced representations of reductive p-adic groups. I. *Ann. Sci. École Norm. Sup. (4)*, 10(4):441–472, 1977.

[6] N. Bourbaki. *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines.* Actualités Scientifiques et Industrielles, No. 1337. Hermann, Paris, 1968.

[7] Alexander Braverman, Michael Finkelberg, and Dennis Gaitsgory. Uhlenbeck spaces via affine Lie algebras. In *The unity of mathematics*, volume 244 of *Progr. Math.*, pages 17–135. Birkhäuser Boston, Boston, MA, 2006.

[8] Alexander Braverman and Dennis Gaitsgory. Crystals via the affine Grassmannian. *Duke Math. J.*, 107(3):561–575, 2001.

[9] Benjamin Brubaker, Daniel Bump, Gautam Chinta, Solomon Friedberg, and Jeffrey Hoffstein. Weyl group multiple Dirichlet series. I. In *Multiple Dirichlet series, automorphic forms, and analytic number theory*, volume 75 of *Proc. Sympos. Pure Math.*, pages 91–114. Amer. Math. Soc., Providence, RI, 2006.

[10] Benjamin Brubaker, Daniel Bump, and Solomon Friedberg. Weyl group multiple Dirichlet series, Eisenstein series and crystal bases. *Ann. of Math. (2)*, to appear.

[11] F. Bruhat and J. Tits. Groupes réductifs sur un corps local. *Inst. Hautes Études Sci. Publ. Math.*, (41):5–251, 1972.

[12] Jean-Luc Brylinski and Pierre Deligne. Central extensions of reductive groups by $\mathbf{K}_2$. *Publ. Math. Inst. Hautes Études Sci.*, (94):5–85, 2001.

[13] Daniel Bump and Maki Nakasuji. Integration on p-adic groups and crystal bases. *Proc. Amer. Math. Soc.*, 138:1595–1605, 2010.

[14] W. Casselman. Introduction to the theory of admissible representations of p-adic reductive groups. *preprint.*

[15] W. Casselman and J. Shalika. The unramified principal series of p-adic groups. II. The Whittaker function. *Compositio Math.*, 41(2):207–231, 1980.

[16] Gautam Chinta and Paul E. Gunnells. Constructing Weyl group multiple Dirichlet series. *J. Amer. Math. Soc.*, 23(1):189–215, 2010.

[17] Gautam Chinta and Omar Offen. A metaplectic Casselman-Shalika formula for GL_r. *preprint.*

[18] Michael Finkelberg and Sergey Lysenko. Twisted geometric Satake equivalence. arXiv:0809.3738.

[19] Benedict H. Gross. On the Satake isomorphism. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 223–237. Cambridge Univ. Press, Cambridge, 1998.

[20] Joel Kamnitzer. Mirkovic-Vilonen cycles and polytopes. *Ann. of Math. (2)*, to appear.

[21] Masaki Kashiwara. On crystal bases. In *Representations of groups (Banff, AB, 1994)*, volume 16 of *CMS Conf. Proc.*, pages 155–197. Amer. Math. Soc., Providence, RI, 1995.

[22] D. A. Kazhdan and S. J. Patterson. Metaplectic forms. *Inst. Hautes Études Sci. Publ. Math.*, (59):35–142, 1984.

[23] D. A. Kazhdan and S. J. Patterson. Towards a generalized Shimura correspondence. *Adv. in Math.*, 60(2):161–234, 1986.

[24] Tomio Kubota. *On automorphic functions and the reciprocity law in a number field.* Lectures in Mathematics, Department of Mathematics, Kyoto University, No. 2. Kinokuniya Book-Store Co. Ltd., Tokyo, 1969.

[25] Hung Yean Loke and Gordon Savin. Modular forms on nonlinear double covers of algebraic groups. *preprint.*

[26] G Lusztig. Canonical bases arising from quantized enveloping algebras. *J. Amer. Math. Soc.*, 3(2):447–498, 1990.

[27] George Lusztig. Introduction to quantized enveloping algebras. In *New developments in Lie theory and their applications (Córdoba, 1989)*, volume 105 of *Progr. Math.*, pages 49–65. Birkhäuser Boston, Boston, MA, 1992.

[28] George Lusztig. *Introduction to quantum groups*, volume 110 of *Progress in Mathematics.* Birkhäuser Boston Inc., Boston, MA, 1993.

[29] George Lusztig. An algebraic-geometric parametrization of the canonical basis. *Adv. Math.*, 120(1):173–190, 1996.

[30] Hideya Matsumoto. Sur les sous-groupes arithmétiques des groupes semi-simples déployés. *Ann. Sci. École Norm. Sup. (4)*, 2:1–62, 1969.

[31] Peter McNamara. Metaplectic Whittaker functions and crystal bases. arXiv:0907.2675.

[32] Peter McNamara. Principal series representations of metaplectic groups over local fields. arXiv:0911.2274.

[33] I. Mirković and K. Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Ann. of Math. (2)*, 166(1):95–143, 2007.

[34] Colette Mœglin and Jean-Loup Waldspurger. *Décomposition spectrale et séries d'Eisenstein*, volume 113 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1994. Une paraphrase de l'Écriture. [A paraphrase of Scripture].

[35] Calvin C. Moore. Group extensions of p-adic and adelic linear groups. *Inst. Hautes Études Sci. Publ. Math.*, (35):157–222, 1968.

[36] Ryan Reich. PhD thesis. Harvard University, 2010.

[37] François Rodier. Décomposition de la série principale des groupes réductifs p-adiques. In *Noncommutative harmonic analysis and Lie groups (Marseille, 1980)*, volume 880 of *Lecture Notes in Math.*, pages 408–424. Springer, Berlin, 1981.

[38] Gordan Savin. Local Shimura correspondence. *Math. Ann.*, 280(2):185–190, 1988.

[39] Gordan Savin. On unramified representations of covering groups. *J. Reine Angew. Math.*, 566:111–134, 2004.

[40] Jean-Pierre Serre. *Corps locaux*. Publications de l'Institut de Mathématique de l'Université de Nancago, VIII. Actualités Sci. Indust., No. 1296. Hermann, Paris, 1962.

[41] Robert Steinberg. Générateurs, relations et revêtements de groupes algébriques. In *Colloq. Théorie des Groupes Algébriques (Bruxelles, 1962)*, pages 113–127. Librairie Universitaire, Louvain, 1962.

[42] Robert Steinberg. *Lectures on Chevalley groups*. Yale University, New Haven, Conn., 1968. Notes prepared by John Faulkner and Robert Wilson.

[43] Martin H. Weissman. Metaplectic tori over local fields. *Pacific J. Math.*, 241(1):169–200, 2009.